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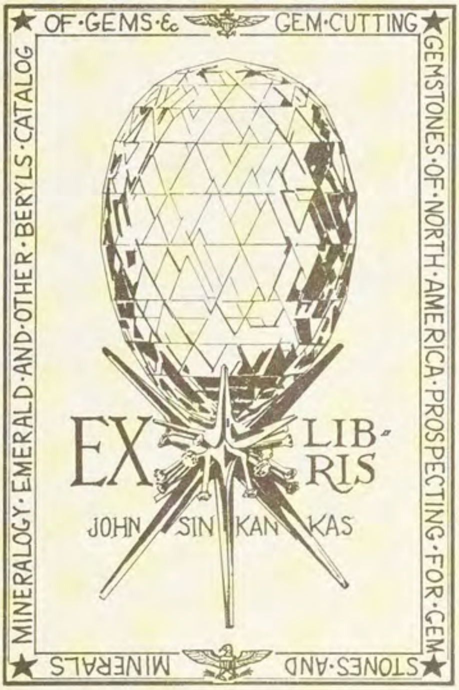

This copy, presented by Miller to W.J. Lewis, himself a noted crystallographer in later years (A Treatise on Crystallography, Cambridge, 1899), was rebound by me (J.S.) using old cloth and paper, April 3, 1979

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Richard Jolly
Oct 1878

A

TREATISE

ON

CRYSTALLOGRAPHY.

BY

W. H. MILLER, M.A., F.R.S., F.G.S., F.C.P.S.,

FELLOW AND TUTOR OF ST JOHN'S COLLEGE,

AND

PROFESSOR OF MINERALOGY IN THE UNIVERSITY OF CAMBRIDGE.

CAMBRIDGE:

PRINTED AT THE PITT PRESS: FOR

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JOHN W. PARKER, WEST STRAND.

M.DCCC.XXXIX.

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W. J. Lewis
from the Author

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ADVERTISEMENT.

The Crystallographic Notation adopted in the following Treatise is taken, with a few unimportant alterations, from Professor Whewell's Memoir "On a general method of calculating the angles of Crystals," printed in the Transactions of the Royal Society for 1825.

The method of indicating the positions of the faces of a Crystal by the points in which radii drawn perpendicular to the faces meet the surface of a sphere, was invented by Professor Neumann of Königsberg (Beiträge zur Krystallonomie) and afterwards, together with the notation, re-invented independently by Grassmann (Zur Krystallonomie und geometrischen Combinationslehre). The use of this method led to the substitution of spherical trigonometry for the processes of solid and analytical geometry in deducing expressions for determining the positions of the faces of crystals and the angles they make with each other. The expressions which in this Treatise have thus been obtained,

are remarkable for their symmetry and simplicity, and are all adapted to logarithmic computation. They are, it is believed, for the most part new. For the convenience of calculation the position of one face with respect to another is represented by the angle between normals to the faces, or by the supplement of the angle between the faces, according to the commonly received definition of the angle between two of the planes that bound a solid.

Arts. 22—24, 28—31, may be omitted by those readers who are satisfied with such a knowledge of the subject as is sufficient for the purpose of finding the elements and the symbols of the faces of a given crystal, or of determining the form and angles of a crystal, having given its elements and the symbols of its faces.

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ERRATA.

Page	Line	
36	8	*after* Let *insert* the.
39	8	from bottom, *insert a comma after* Octahedron.
39	6	from bottom, for $p'e'$ read p', e'.
50	2	*for* {4 1 0} *read* {4 3 0}.
81	19	*for* (182) *read* (183).
86	6	after o, x, p, t *insert* belong.
86	8	*for* wanting *read* want.
87	5	from bottom, *for* 87 *read* 88.
88	7	from bottom, *for* κ *read* σ.
92	11	for g read q.
107	10	from bottom, *for* Scheelale *read* Scheelate.

114 At the end of Art. 266, *add* The distance of the crystal from the eye of the observer should be from one inch to two inches; the image of A formed by reflexion at either of the faces p, q will then be seen clearly, while the crystal itself remains unseen. It is essential that the signal B and the image of A should be seen distinctly. When the distance of distinct vision of the observer's eye differs from AC or BC, the eye must be armed with a lens, or a small telescope having a power of from one to three, capable of shewing the signals distinctly.

CRYSTALLOGRAPHY.

CHAPTER I.

GENERAL GEOMETRICAL PROPERTIES OF CRYSTALS.

1. MANY natural substances and many of the results of chemical operations occur in the form of solids which are bounded by plane surfaces, and which commonly exhibit a tendency to separate, when broken, in the directions of planes, passing through any point within them, either parallel to some of their bounding planes, or else making given angles with them.

Solids of this description are called crystals. The planes by which they are bounded are called their faces, and the planes in which they have a tendency to separate, their cleavage planes.

2. The mutual inclinations of the faces and cleavage planes of a crystal are subject to a law which we now proceed to enunciate.

Through any point O (fig. 1) within the crystal, let planes be drawn parallel to each of its faces and cleavage planes. Let OX, OY, OZ be any three intersections of those planes, not all in one plane; and let any face or cleavage plane meet OX, OY, OZ in A, B, C. Then, if any other face or cleavage plane meet OX, OY, OZ in H, K, L, and if we consider HO, KO, LO to be positive or negative according as they are measured in the same directions as AO, BO, CO, or in different directions, it will be found that

$$\frac{1}{h}\frac{AO}{HO}=\frac{1}{k}\frac{BO}{KO}=\frac{1}{l}\frac{CO}{LO},$$

where h, k, l, may be any positive or negative whole numbers, one or two of which may be zero

When one of the numbers h, k, l becomes zero, one of the distances HO, KO, LO corresponding, becomes indefinitely great, and, therefore, is parallel to the line along which that distance is measured.

3. Since the positions of the faces and cleavage planes are subject to the same law, it follows, that a cleavage plane is always parallel to a plane which either is, or may be, a face of the crystal. This being the case, in speaking of the faces of a crystal, we shall always suppose the cleavage planes to be included.

4. OX, OY, OZ are called the axes of the crystal, O their origin; AO, BO, CO, or any three lines in the same ratio, the parameters of the crystal; h, k, l the indices of the face HKL. This face will be denoted by the symbol $(h\,k\,l)$, a negative index being distinguished by a minus sign placed over it.

5. The indices h, k, l, which by taking different integral values determine the positions of the different faces of the same crystal, are seldom large. When the axes and parameters are properly chosen the highest index does not commonly exceed six.

The inclinations of the axes and the ratios of the parameters are the same, at a given temperature, for all crystals of the same species. The symbols of the faces may be different. Hence, the angles YOZ, ZOX, XOY, which the axes make with each other, and the ratios of two of the parameters AO, BO, CO to the third, are five elements by which each crystalline species is characterized.

6. It will be sufficient for the present if we suppose the law stated in (2) to hold when the axes are three given lines in which the planes drawn through a point within the crystal, parallel to its faces, intersect each other, and the parameters are the portions of the axes cut off by a given face. Hereafter we shall prove that, if through a point within the crystal, planes be drawn parallel to all the

possible faces of the crystal, and that the law stated in (2) hold when the axes are three given intersections of those planes, and the parameters are the portions of the axes cut off by a given face; it will also hold, when any three intersections are taken for axes, and the portions of them cut off by any plane are taken for parameters.

7. The law enunciated in (2) may be put, as follows, under a different form, which, though not quite so simple, presents a clearer view of the relative positions of the faces of a crystal.

Let OX, OY, OZ (fig. 2) be the axes of a crystal, a, b, c its parameters.

In OX take $OA=a$, $OA_2=\frac{1}{2}a$, $OA_3=\frac{1}{3}a$, ... towards X; $OA_{-1}=a$, $OA_{-2}=\frac{1}{2}a$, $OA_{-3}=\frac{1}{3}a$, ... in the opposite direction, and $OA_0=\frac{1}{0}a=\infty$ in either direction. And let the points B, B_{-1}, B_2, B_{-2}, B_3, B_{-3}, ..., B_0, C, C_{-1}, C_2, C_{-2}, C_3, C_{-3}, ..., C_0, be determined in the same manner from b and c respectively. Then, any face of the crystal will be parallel to a plane drawn through three points thus determined, one being taken in each of the three axes.

For if the face $(h\,k\,l)$ meet the axes OX, OY, OZ in H, K, L. Then (2)

$$\frac{1}{h}\frac{OA}{OH}=\frac{1}{k}\frac{OB}{OK}=\frac{1}{l}\frac{OC}{OL}.$$

But, according to the notation we have adopted,

$$OA_h=\frac{1}{h}a,\quad OB_k=\frac{1}{k}b,\quad OC_l=\frac{1}{l}c,$$

these distances being measured towards X, Y, Z, or in the opposite directions, according as the corresponding indices h, k, l, are positive or negative;

$$\therefore\ \frac{OA_h}{OH}=\frac{OB_k}{OK}=\frac{OC_l}{OL};$$

$\therefore$ the face $(h\,k\,l)$ is parallel to the plane passing through the three points A_h, B_k, C_l.

8. To find the ratios of the cosines of the angles which a perpendicular to the face $(h\,k\,l)$ makes with the axes of the crystal, in terms of the indices of the face and the parameters of the crystal.

Let the axes OX, OY, OZ, (fig. 3) meet the surface of a sphere described round O as a center in X, Y, Z, and let OP, drawn perpendicular to the face HKL, the symbol of which is $(h\,k\,l)$, meet HKL in p, and the surface of the sphere in P. Then,

$$\frac{Op}{HO} = \cos PX, \quad \frac{Op}{KO} = \cos PY, \quad \frac{Op}{LO} = \cos PZ.$$

Therefore, substituting these values of HO, KO, LO in (2), if $AO = a$, $BO = b$, $CO = c$, we have

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ.$$

In all problems of Crystallography which may hereafter present themselves, we shall refer the faces of crystals to the surface of a sphere by means of radii drawn perpendicular to the faces, and all our calculations will be performed by spherical trigonometry applied to expressions deduced from the above equations.

9. The sphere to the surface of which the faces of a crystal are referred, will be called the sphere of projection. The extremity of a radius of the sphere drawn perpendicular to any face, will be called the pole of that face. A face and its pole will be usually denoted by the same letter and by the same symbol. The points in which the axes of the crystal meet the surface of the sphere of projection will be invariably denoted by X, Y, Z.

10. Let the axes of any crystal meet the surface of the sphere of projection in X, Y, Z, (fig. 4). Let a, b, c be the parameters of the crystal, ABC the polar triangle of XYZ. Then, AY, AZ are quadrants, $\therefore \cos AY = 0$, $\cos AZ = 0$;

$$\therefore \frac{a}{1} \cos AX = \frac{b}{0} \cos AY = \frac{c}{0} \cos AZ;$$

$\therefore$ (8) A is the pole of (1 0 0). Similarly, B is the pole of (0 1 0), and C is the pole of (0 0 1).

11. Let P be the pole of the face $(h\, k\, l)$. Then, (8)

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ.$$

When P and A are on the same side of the great circle BC, PX is less than a quadrant, therefore $\cos PX$ is positive. When P and A are on opposite sides of BC, PX is greater than a quadrant, therefore $\cos PX$ is negative. Hence, if we assume h to be positive in the former case, it will be negative in the latter. In like manner k will be positive or negative according as P and B are on the same or opposite sides of CA. And l will be positive or negative according as P and C are on the same or opposite sides of AB.

When P is in the great circle BC, PX is a quadrant, therefore $\cos PX = 0$; therefore $h = 0$. When P is in CA, $\cos PY = 0$, therefore $k = 0$. When P is in AB, $\cos PZ = 0$, therefore $l = 0$.

Hence, if diameters AA', BB', CC', PP' be drawn, the symbols of the points A, B, C, P being $A\,(1\,0\,0)$, $B\,(0\,1\,0)$, $C\,(0\,0\,1)$, $P\,(h\,k\,l)$, those of A', B', C', P' will be $A'\,(\bar{1}\,0\,0)$, $B'\,(0\,\bar{1}\,0)$, $C'\,(0\,0\,\bar{1})$, $P'\,(\bar{h}\,\bar{k}\,\bar{l})$.

12. X, Y, Z (fig. 5.) are any three points on the surface of a sphere; P, Q, R any three points in a great circle; to find the relation between the distances of P, Q, R from each of the points X, Y, Z.

From the spherical triangles PQX, RQX, we have

$$\cos PX = \cos QX \cos PQ + \sin QX \sin PQ \cos PQX,$$
$$\cos RX = \cos QX \cos QR + \sin QX \sin QR \cos RQX.$$

Multiply the first equation by $\sin QR$, the second by $\sin PQ$, and add, observing that

$$\cos PQX + \cos RQX = 0, \quad \text{and that}$$
$$\sin QR \cos PQ + \cos QR \sin PQ = \sin PR.$$

The equation thus obtained, and two others deducible from it by writing Y and Z successively in the place of X, are

$$\cos PX \sin QR + \cos RX \sin PQ = \cos QX \sin PR,$$
$$\cos PY \sin QR + \cos RY \sin PQ = \cos QY \sin PR,$$
$$\cos PZ \sin QR + \cos RZ \sin PQ = \cos QZ \sin PR.$$

Whence, eliminating $\sin PQ$, $\sin PR$, $\sin QR$, successively,

$$\frac{1}{\sin PQ}[\cos PX \cos QY - \cos PY \cos QX]$$
$$= \frac{1}{\sin PR}[\cos PX \cos RY - \cos PY \cos RX]$$
$$= \frac{1}{\sin QR}[\cos QX \cos RY - \cos QY \cos RX],$$
$$\frac{1}{\sin PQ}[\cos PZ \cos QX - \cos PX \cos QZ]$$
$$= \frac{1}{\sin PR}[\cos PZ \cos RX - \cos PX \cos RZ]$$
$$= \frac{1}{\sin QR}[\cos QZ \cos RX - \cos QX \cos RZ],$$
$$\frac{1}{\sin PQ}[\cos PY \cos QZ - \cos PZ \cos QY]$$
$$= \frac{1}{\sin PR}[\cos PY \cos RZ - \cos PZ \cos RY]$$
$$= \frac{1}{\sin QR}[\cos QY \cos RZ - \cos QZ \cos RY]$$

Eliminating two of the quantities sin PQ, sin PR, sin QR, between two of the preceding equations, we obtain

$$(\cos PY \cos RZ - \cos PZ \cos RY) \cos QX$$
$$+ (\cos PZ \cos RX - \cos PX \cos RZ) \cos QY$$
$$+ (\cos PX \cos RY - \cos PY \cos RX) \cos QZ = 0.$$

13. Let X, Y, Z be the points in which the axes of a crystal meet the surface of the sphere of projection; P, R the poles $(h\,k\,l)$, $(p\,q\,r)$; a, b, c the parameters of the crystal. Then (8)

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ,$$

$$\frac{a}{p} \cos RX = \frac{b}{q} \cos RY = \frac{c}{r} \cos RZ.$$

Therefore eliminating cos PX, cos PY, cos PZ, cos RX, cos RY, cos RZ between the above equations and the equation at the end of (12), we have

$$\mathrm{u}a \cos QX + \mathrm{v}b \cos QY + \mathrm{w}c \cos QZ = 0,$$

where

$$\mathrm{u} = kr - lq, \quad \mathrm{v} = lp - hr, \quad \mathrm{w} = hq - kp.$$

14. The great circle passing through the poles $(h\,k\,l)$, $(p\,q\,r)$, may be denoted by the symbol [u v w], where u, v, w have the values assigned to them in (13). Since the poles P, R may be denoted by the indices h, k, l; p, q, r, or by any numbers proportional to h, k, l; p, q, r respectively, it follows, that the great circle PR may be denoted by any three numbers proportional to u, v, w. When u, v, w have a common measure, it will be found convenient to employ as indices, the lowest whole numbers in the required ratio.

15. Let [h k l], [p q r] be the symbols of two great circles, each of which passes through the poles of any two faces not parallel to each other; and let the great circle

[h k l] meet the great circle [p q r] in the point Q. Then, since Q is a point in each of the great circles, (13)

$$\mathrm{h}a \cos QX + \mathrm{k}b \cos QY + \mathrm{l}c \cos QZ = 0,$$

$$\mathrm{p}a \cos QX + \mathrm{q}b \cos QY + \mathrm{r}c \cos QZ = 0.$$

Whence, eliminating $\cos QX$, $\cos QY$, $\cos QZ$ successively, we obtain

$$\frac{a}{u} \cos QX = \frac{b}{v} \cos QY = \frac{c}{w} \cos QZ,$$

where

$$u = \mathrm{kr} - \mathrm{lq}, \quad v = \mathrm{lp} - \mathrm{hr}, \quad w = \mathrm{hq} - \mathrm{kp}.$$

The indices h, k, l, p, q, r are integers; $\therefore$ u, v, w are integers, and therefore (8) Q is the pole of the face $(u\,v\,w)$. Hence, it appears that a face may always exist having its pole in the intersection of any two great circles, each of which passes through the poles of any two faces not parallel to each other.

16. When three or more faces of a crystal have their poles in the same great circle, they are said to form a zone. The great circle passing through the poles of any two faces not parallel to each other, and which, therefore, passes through the pole of any other face in the same zone with them, will be called a zone-circle. The diameter which joins its poles will be called the axis of the zone. A zone and its zone-circle will be denoted by the same symbol.

17. It appears from (13) that if [u v w] be the symbol of the zone containing the faces $(h\,k\,l)$, $(p\,q\,r)$,

$$\mathrm{u} = kr - lq, \quad \mathrm{v} = lp - hr, \quad \mathrm{w} = hq - kp,$$

and from (13) that if $(u\,v\,w)$ be the symbol of the face common to the zones [h k l], [p q r],

$$u = \mathrm{kr} - \mathrm{lq}, \quad v = \mathrm{lp} - \mathrm{hr}, \quad w = \mathrm{hq} - \mathrm{kp},$$

or, that the expressions for u, v, w, in terms of *h*, *k*, *l*, *p*, *q*, *r*, are precisely similar to the expressions for *u*, *v*, *w*, in terms of h, k, l, p, q, r.

We shall sometimes find it convenient to denote the zone containing the faces (*h k l*), (*p q r*), by the symbol [*h k l*, *p q r*], and the face common to the zones [h k l], [p q r], by the symbol (h k l, p q r).

18. The intersections of the faces of a zone, or of the faces produced, are parallel to the axis of the zone, and therefore, to one another. In many cases the parallelisms of the edges resulting from the intersections of a series of faces belonging to the same zone can be ascertained by simple inspection. The method of determining by observation whether a face does or does not belong to a zone containing two given faces, when it does not meet them, or when the edges it makes with them are so short that their parallelism is doubtful, will be described when we come to explain the use of Wollaston's Goniometer.

When, by observing the parallelism of the edges or otherwise, it has been ascertained that a face is in the same zone with two given faces, and that it is also in the same zone with two other given faces, the symbols of the two zones, and, from these, the symbol of the face which is common to them, may be found by the methods of (14) and (15).

19. The points in which any two zone-circles intersect are the opposite extremities of a diameter of the sphere of projection. Hence, (11) the symbol of one point of intersection is obtained from that of the other by merely changing the signs of all the indices.

20. If [u v w] be the symbol of the zone-circle through the poles (*h k l*), (*p q r*), it is easily seen that [$\bar{\mathrm{u}}$ v w] will be the symbol of the zone-circle through the poles ($\bar{h}\,k\,l$), ($\bar{p}\,q\,r$); and also that if the zone-circles [h k l], [p q r] intersect (*u v w*), the zone-circles [$\bar{\mathrm{h}}$ k l], [$\bar{\mathrm{p}}$ q r] will intersect

in $(\bar{u}\,v\,w)$. Hence, if the zone-circles $[h\,k\,l,\ p\,q\,r]$, $[h'\,k'\,l',\ p'\,q'\,r']$, intersect in $(u\,v\,w)$, the zone-circles $[\bar{h}\,k\,l,\ \bar{p}\,q\,r]$, $[\bar{h}'\,k'\,l',\ \bar{p}'\,q'\,r']$ will intersect in $(\bar{u}\,v\,w)$.

If the zone-circles $[h\,k\,l,\ p\,q\,r]$, $[h'\,k'\,l',\ p'\,q'\,r']$ intersect in $(u\,v\,w)$, it is manifest that $[l\,h\,k,\ r\,p\,q]$, $[l'\,h'\,k',\ r'\,p'\,q']$ intersect in $(w\,u\,v)$, and that $[h\,l\,k,\ p\,r\,q]$, $[h'\,l'\,k',\ p'\,r'\,q']$ intersect in $(u\,w\,v)$.

21. Let Q be the pole of a face $(u\,v\,w)$, in the zone $[\mathrm{u\,v\,w}]$. Then, (13), (8)

$$\mathrm{u}\,a\cos QX + \mathrm{v}\,b\cos QY + \mathrm{w}\,c\cos QZ = 0,$$

$$\frac{a}{u}\cos QX = \frac{b}{v}\cos QY = \frac{c}{w}\cos QZ\,;$$

$$\therefore\ \mathrm{u}u + \mathrm{v}v + \mathrm{w}w = 0.$$

This equation expresses the condition that the face $(u\,v\,w)$ may be in the zone $[\mathrm{u\,v\,w}]$. Any whole numbers which, when substituted for u, v, w, satisfy the above equation, are the indices of a face in the zone $[\mathrm{u\,v\,w}]$; and any three whole numbers which, when substituted for u, v, w, satisfy the same equation, are the indices of a zone containing the face $(u\,v\,w)$.

22. When the zone-circle $[\mathrm{u\,v\,w}]$ passes through the pole $(u\,v\,w)$, (21)

$$\mathrm{u}u + \mathrm{v}v + \mathrm{w}w = 0.$$

Hence, in order to find the poles which lie in a given zone-circle, or the zone-circles passing through a given pole, we must find the integral values (one or two of which may be zero), of x, y, z, which satisfy the equation

$$ax + by + cz = 0\,;$$

where a, b, c are the indices of the given zone-circle in the former case, and of the given pole in the latter.

Let the coefficients c, b be prime to each other. Transform $c \div b$ into a continued fraction, and let $c' \div b'$ be the

last but one of the resulting converging fractions. Then, by the rule for solving indeterminate equations of the first degree, $y = \pm (c'ax - mc)$, $z = \pm (mb - b'ax)$; where the upper or lower sign is to be taken, according as cb' is greater or less than bc'. The value of x being assumed, the corresponding values of y and z may be obtained by substituting different positive or negative whole numbers for m.

23. If the zone-circle [h k l] passing through the poles $(h\,k\,l)$, $(h'\,k'\,l')$, intersect the zone-circle [p q r] passing through the poles $(p\,q\,r)$, $(p'q'r')$, in the pole $(u\,v\,w)$; values of h, k, l, h', k', l', p, q, r, p', q', r', can always be found, such that the three indices of each pole shall be severally numerically less than the indices u, v, w, or not greater than unity.

The values of $h, k, l,\ h', k', l',\ p, q, r,\ p', q', r'$ must satisfy the equations

$$u\mathrm{h} + v\mathrm{k} + w\mathrm{l} = 0, \quad u\mathrm{p} + v\mathrm{q} + w\mathrm{r} = 0,$$
$$\mathrm{h}h + \mathrm{k}k + \mathrm{l}l = 0, \quad \mathrm{p}p + \mathrm{q}q + \mathrm{r}r = 0,$$
$$\mathrm{h}h' + \mathrm{k}k' + \mathrm{l}l' = 0, \quad \mathrm{p}p' + \mathrm{q}q' + \mathrm{r}r' = 0.$$

We have, therefore, to shew that if a, b, c be any whole numbers, the equation $ax + by + cz = 0$, may be satisfied by two sets of integral values of x, y, z, such that, if either of them be α, β, γ, the equation $\alpha x + \beta y + \gamma z = 0$, may be satisfied by two sets of integral values of x, y, z which are severally numerically less than a, b, c, or not greater than unity. The most unfavourable case is that in which the largest of the three numbers a, b, c is prime to each of the other two, and no two are equal. Let c be greater than b, and b greater than a. Since a is less than b, x may have a value which is either unity or some number less than a, and which makes ax numerically less than by. $\pm y = (c'ax - mc)$, therefore y may be made less than c, and the signs of ax, by different. In which case, since $-cz = by + ax$, z will be less than b, and the sign of cz different from that of by, and therefore the same as that of ax. Also, since $\pm z = (mb - b'ax)$,

with either the same value of x, or a different value, which is either unity or a number less than a, a value of z may be found less than b, and making the signs of ax, cz different. And therefore the corresponding value of y will be less than c. Hence, if either of the above values of x, y, z be α, β, γ respectively, α will be less than a, or unity, β less than c, and γ less than b. In like manner, the equation $\alpha x+\beta y+\gamma z=0$, may be satisfied by two sets of values of x, v, z less than α, β, γ, or not greater than unity, and therefore less than a, b, c, or not greater than unity. Hence values of h, k, l, h', k', l', p, q, r, p', q', r', can always be found capable of satisfying the equations by which they are connected with u, v, w, such that the three indices of each pole shall be lower numbers than u, v, w, or not greater than unity.

24. It appears from the preceding investigation, that the pole of any face $(u\,v\,w)$ is the intersection of two zone-circles, each of which passes through the poles of two faces having indices more simple than those of $(u\,v\,w)$. In the same manner, the poles of each of these faces is the intersection of two zone-circles passing through the poles of faces having indices still more simple, and so on, till at last we arrive at the poles of four faces the indices of which are either unity or zero.

25. Having given the distances between four poles in the same zone-circle, and the symbols of three of them; to find the symbol of the fourth.

Let P, Q, R, S, (fig. 6), be the four poles in one zone-circle, and let their symbols be $P\,(e\,f\,g)$, $Q\,(h\,k\,l)$, $R\,(p\,q\,r)$, $S\,(u\,v\,w)$; X, Y, Z the extremities of radii drawn parallel to the axes of the crystal; a, b, c the parameters of the crystal. Then,

$$\frac{a}{e}\cos PX = \frac{b}{f}\cos PY = \frac{c}{g}\cos PZ,$$

$$\frac{a}{h}\cos QX = \frac{b}{k}\cos QY = \frac{c}{l}\cos QZ,$$

$$\frac{a}{p}\cos RX = \frac{b}{q}\cos RY = \frac{c}{r}\cos RZ,$$

$$\frac{a}{u}\cos SX = \frac{b}{v}\cos SY = \frac{c}{w}\cos SZ.$$

P, Q, R are in the same great circle, and PQ is less than PR, therefore, (12),

$$\frac{1}{\sin PQ}[\cos PX \cos QY - \cos PY \cos QX]$$

$$= \frac{1}{\sin QR}[\cos QX \cos RY - \cos QY \cos RX],$$

$$\frac{1}{\sin PQ}[\cos PZ \cos QX - \cos PX \cos QZ]$$

$$= \frac{1}{\sin QR}[\cos QZ \cos RX - \cos QX \cos RZ],$$

$$\frac{1}{\sin PQ}[\cos PY \cos QZ - \cos PZ \cos QY]$$

$$= \frac{1}{\sin QR}[\cos QY \cos RZ - \cos QZ \cos RY].$$

P, S, R are in the same zone-circle, and PS is less than PR, therefore, (12),

$$\frac{1}{\sin PS}[\cos PX \cos SY - \cos PY \cos SX]$$

$$= \frac{1}{\sin SR}[\cos SX \cos RY - \cos SY \cos RX],$$

$$\frac{1}{\sin PS}[\cos PZ \cos SX - \cos PX \cos SZ]$$

$$= \frac{1}{\sin SR}[\cos SZ \cos RX - \cos SX \cos RZ],$$

$$\frac{1}{\sin PS}[\cos PY \cos SZ - \cos PZ \cos SY]$$

$$= \frac{1}{\sin PS}[\cos SY \cos RZ - \cos SZ \cos RY].$$

Whence, dividing each of the equations between $\sin PS$, $\sin SR$, by the corresponding equation between $\sin PQ$,

let $\lambda\mu\rho$ be any pole k & let $[efg]$ $[pqr]$ be zone circles kP, kR

then $\left.\begin{aligned} e &= f\rho - g\mu \\ f &= g\lambda - e\rho \\ g &= e\mu - f\lambda \end{aligned}\right\}$ $\left.\begin{aligned} p &= q\rho - r\mu \\ q &= r\lambda - p\rho \\ r &= p\mu - q\lambda \end{aligned}\right\}$ see Tract. p. 10

$\sin QR$, and substituting for the ratios of the cosines the values given above,

$$\frac{[P, Q]}{\sin PQ}\,\frac{[S, R]}{\sin SR} = \frac{[Q, R]}{\sin QR}\,\frac{[P, S]}{\sin PS},$$

where

$$\frac{[P, Q]}{[Q, R]} = \frac{fl - gk}{kr - lq}\times\lambda = \frac{gh - el}{lp - hr}\times\mu = \frac{ek - fh}{hq - kp}\times\rho, = \frac{-(he + kf + lg)}{hp + kq + lr}$$

$$\frac{[P, S]}{[S, R]} = \frac{fw - gv}{vr - wq}\times\lambda = \frac{gu - ew}{wp - ur}\times\mu = \frac{ev - fu}{uq - vp}.\times\rho = \frac{-(ue + vf + wg)}{up + qv + ru}$$

Sin PQ sin SR, sin QR sin PS must be to each other as some whole numbers. Let

$$\frac{\sin PQ}{\sin QR}\,\frac{\sin SR}{\sin PS} = \frac{m}{n}; \quad \therefore \quad \frac{[S, R]}{[P, S]} = \frac{m}{n}\,\frac{[Q, R]}{[P, Q]}.$$

Whence,

$$u = me[Q, R] + np[P, Q],$$

$$v = mf[Q, R] + nq[P, Q],$$

$$w = mg[Q, R] + nr[P, Q].$$

When PS is greater than PR, we have

$$\frac{[P, Q]}{\sin PQ}\,\frac{[R, S]}{\sin RS} = \frac{[Q, R]}{\sin QR}\,\frac{[P, S]}{\sin PS},$$

where

$$\frac{[P, S]}{[R, S]} = \frac{fw - gv}{qw - rv} = \frac{gu - ew}{ru - pw} = \frac{ev - fu}{pv - qu}.$$

Whence

$$u = me[Q, R] - np[P, Q],$$

$$v = mf[Q, R] - nq[P, Q],$$

$$w = mg[Q, R] - nr[P, Q].$$

26. Having given the symbols of four poles in the same zone-circle, and the distances between three of them; to find the fourth.

Let

$$\tan\theta = \frac{[P, Q]}{[Q, R]}\,\frac{[S, R]}{[P, S]}\,\frac{\sin QR}{\sin PQ}, \qquad \therefore\quad \frac{\sin SR}{\sin PS} = \tan\theta,$$

$$\therefore\quad \tan(PS - \tfrac{1}{2}PR) = \tan\tfrac{1}{2}PR\tan\left(\frac{\pi}{4} - \theta\right).$$

27. $$\operatorname{Sin} QR = \sin PR \sin PQ[\cot PQ - \cot PR],$$

$$\sin SR = \sin PR \sin PS[\cot PS - \cot PR].$$

Therefore

$$\frac{\cot PS - \cot PR}{\cot PQ - \cot PR} = \frac{[P, Q]}{[Q, R]}\,\frac{[S, R]}{[P, S]}.$$

28. The axes of any three possible zones, not being in one plane, may be employed as crystallographic axes.

Let X, Y, Z (fig. 7) be the points in which the axes of the crystal meet the surface of the sphere of projection; a, b, c the parameters of the crystal; X', Y', Z' the poles of any three zone-circles intersecting in A, B, C; P the pole of any face of the crystal; M the intersection of CA, BP; N the intersection of AB, CP. Let the symbols of the poles be $A(efg)$, $B(hkl)$, $C(pqr)$, $P(uvw)$, $M(\lambda\mu\nu)$, $N(\pi\rho\sigma)$. Then (14), (15),

$$\lambda = (re - pg)(hv - ku) - (pf - qe)(lu - hw),$$

$$\nu = (qg - rf)(lu - hw) - (re - pg)(kw - lv),$$

$$\pi = (gh - el)(pv - qu) - (ek - fh)(ru - pw),$$

$$\rho = (ek - fh)(qw - rv) - (fl - gk)(pv - qu).$$

ANB, AMC are great circles; therefore, writing X', Y', Z' for X, Y, Z in (12), and dividing the equations containing X', Y', Z' by the corresponding equations containing X, Y, Z, we have

$$\frac{\cos AX' \cos NY' - \cos AY' \cos NX'}{\cos AX \cos NY - \cos AY \cos NX}$$

$$= \frac{\cos BY' \cos NX' - \cos BX' \cos NY'}{\cos BY \cos NX - \cos BX \cos NY},$$

$$\frac{\cos AX' \cos MZ' - \cos AZ' \cos MX'}{\cos AX \cos MZ - \cos AZ \cos MX}$$

$$= \frac{\cos CZ' \cos MX' - \cos CX' \cos MZ'}{\cos CZ \cos MX - \cos CX \cos MZ}.$$

But

$$\frac{a}{e} \cos AX = \frac{b}{f} \cos AY = \frac{c}{g} \cos AZ,$$

$$\frac{a}{h} \cos BX = \frac{b}{k} \cos BY = \frac{c}{l} \cos BZ,$$

$$\frac{a}{p} \cos CX = \frac{b}{q} \cos CY = \frac{c}{r} \cos CZ,$$

$$\frac{a}{\lambda} \cos MX = \frac{b}{\mu} \cos MY = \frac{c}{\nu} \cos MZ,$$

$$\frac{a}{\pi} \cos NX = \frac{b}{\rho} \cos NY = \frac{c}{\sigma} \cos NZ,$$

$$\cos AY' = 0, \ \cos AZ' = 0, \ \cos BX' = 0,$$

$$\cos CX' = 0, \ \cos MY' = 0, \ \cos NZ' = 0.$$

Also

$$\frac{\cos MX'}{\cos PX'} = \frac{\sin MB}{\sin PB} = \frac{\cos MZ'}{\cos PZ'}, \quad \frac{\cos NX'}{\cos PX'} = \frac{\sin NC}{\sin PC} = \frac{\cos NY'}{\cos PY'};$$

$$\therefore \ \frac{e(k\pi - h\rho) \cos AX'}{\cos AX \cos PX'} = \frac{h(e\rho - f\pi) \cos BY'}{\cos BX \cos PY'},$$

$$\frac{e(r\lambda - p\nu) \cos AX'}{\cos AX \cos PX'} = \frac{p(e\nu - g\lambda) \cos CZ'}{\cos CX \cos PZ'},$$

$$k\pi - h\rho = (fh - ek)\,[\mathrm{e}u + \mathrm{f}v + \mathrm{g}w],$$
$$e\rho - f\pi = (fh - ek)\,[\mathrm{h}u + \mathrm{k}v + \mathrm{l}w],$$
$$r\lambda - p\nu = (pg - re)\,[\mathrm{e}u + \mathrm{f}v + \mathrm{g}w],$$
$$e\nu - g\lambda = (pg - re)\,[\mathrm{p}u + \mathrm{q}v + \mathrm{r}w].$$

Where

$$\mathrm{e} = kr - lq, \quad \mathrm{f} = lp - hr, \quad \mathrm{g} = hq - kp,$$
$$\mathrm{h} = qg - rf, \quad \mathrm{k} = re - pg, \quad \mathrm{l} = pf - qe,$$
$$\mathrm{p} = fl - gk, \quad \mathrm{q} = gh - el, \quad \mathrm{r} = ek - fh.$$

Hence

$$\frac{a'}{u'} \cos PX' = \frac{b'}{v'} \cos PY' = \frac{c'}{w'} \cos PZ',$$

where a', b', c' depend only upon the angles between the old and new axes, and

$$u' = \mathrm{e}u + \mathrm{f}v + \mathrm{g}w,$$
$$v' = \mathrm{h}u + \mathrm{k}v + \mathrm{l}w,$$
$$w' = \mathrm{p}u + \mathrm{q}v + \mathrm{r}w.$$

u', v', w' are whole numbers, and therefore OX', OY', OZ' may be taken for crystallographic axes.

The coefficients of u, v, w, in the expressions for u', v', w', are the indices of the zone-circles BC, CA, AB, their symbols being BC [e f g], CA [h k l], AB [p q r].

29. To change the parameters of a crystal.

Let $(h\,k\,l)$ be the symbol of a face P, with parameters a, b, c; $(h'\,k'\,l')$ the symbol of P, when referred to the same axes, with parameters a', b', c'.

Then

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ,$$

and

$$\frac{a'}{h'} \cos PX = \frac{b'}{k'} \cos PY = \frac{c'}{l'} \cos PZ\,;$$

$$\therefore\ \frac{a'}{h'} = \frac{a}{h},\quad \frac{b'}{k'} = \frac{b}{k},\quad \frac{c'}{l'} = \frac{c}{l}.$$

30. P, Q, R, S are four poles in the same zone-circle. When they are referred to a system of axes meeting the surface of the sphere of projection in X, Y, Z, their symbols are $P(efg)$, $Q(hkl)$ $R(pqr)$, $S(uvw)$. When they are referred to a system of axes meeting the surface of the sphere of projection in X', Y', Z', their symbols are $P(e'f'g')$, $Q(h'k'l')$, $R(p'q'r')$, $S(u'v'w')$. If

$$\frac{[P, Q]}{[Q, R]} = \frac{fl - gk}{kr - lq} = \frac{gh - el}{lp - hr} = \frac{ek - fh}{hq - kp},$$

$$\frac{[P, S]}{[S, R]} = \frac{fw - gv}{vr - wq} = \frac{gu - ew}{wp - ur} = \frac{ev - fu}{uq - vp};$$

we have (25)

$$\frac{[P, Q]}{\sin PQ}\,\frac{[S, R]}{\sin SR} = \frac{[Q, R]}{\sin QR}\,\frac{[P, S]}{\sin PS}.$$

And if

$$\frac{[P', Q']}{[P', R']} = \frac{f'l' - g'k'}{k'r' - l'q'} = \frac{g'h' - e'l'}{l'p' - h'r'} = \frac{e'k' - f'h'}{h'q' - k'p'},$$

$$\frac{[P', S']}{[S', R']} = \frac{f'w' - g'v'}{v'r' - w'q'} = \frac{g'u' - e'w'}{w'p' - u'r'} = \frac{e'v' - g'u'}{u'q' - v'p'},$$

we have

$$\frac{[P', Q']}{\sin PQ}\,\frac{[S', R']}{\sin SR} = \frac{[Q', R']}{\sin QR}\,\frac{[P', S']}{\sin PS};$$

$$\therefore \quad \frac{[P', Q']}{[P, Q]}\,\frac{[S', R']}{[S, R]} = \frac{[Q', R']}{[Q, R]}\,\frac{[P', S']}{[P, S]}.$$

Let

$$\frac{[Q, R]}{[P, Q]}\,\frac{[P, S]}{[S, R]}\,\frac{[P', Q']}{[Q', R']} = \frac{m}{n}; \quad \therefore \quad \frac{[P', S']}{[S', R']} = \frac{m}{n},$$

whence

$$u' = e'n + p'm, \quad v' = f'n + q'm, \quad w' = g'n + r'm.$$

31. Having given the symbols of four poles P, Q, R, S, (fig. 8.) when referred to each of two systems of axes OX, OY, OZ; OX', OY', OZ', and the symbol of any other pole referred to the axes OX, OY, OZ; to find its symbol when referred to OX', OY', OZ'.

Let P, Q, R, S be the poles; ABC the polar triangle of $X'\,Y'\,Z'$. Therefore the symbols of A, B, C, when referred to OX', OY', OZ', will be $A(1\,0\,0)$, $B(0\,1\,0)$, $C(0\,0\,1)$. Let PQ, RS meet each other in T, and let them meet BC in U, V. The symbol of T may be found when referred to OX, OY, OZ and also when referred to OX', OY', OZ'; and the symbols of U, V may be found when referred to OX', OY', OZ'. The symbols of P, Q, T when referred to OX, OY, OZ, and to OX', OY', OZ', and the symbol of U when referred to OX', OY', OZ', being known, the symbol of U may be found when referred to OX, OY, OZ. In like manner may be found the symbol of V when referred to OX, OY, OZ. Hence may be found the symbol of the zone-circle BC when referred to OX, OY, OZ. In like manner may be found the symbols of CA, AB referred to OX, OY, OZ. Hence, the symbols of the zone-circles being known, the symbol of any pole when referred to OX', OY', OZ' may be found by (28).

32. In many crystals axes may be discovered which make right angles with each other; in others, axes of which one is perpendicular to the other two, and in others axes making equal angles with each other. In the crystals with equiangular axes, and in some of the crystals with rectangular axes, equal parameters may be found, and, among the remaining crystals with rectangular axes, some which have two of the parameters equal. Upon the differences in the positions of the axes with respect to each other, and in the relation between the parameters, above enumerated, is founded the arrangement of crystals in systems.

1. In the Octahedral system the axes are rectangular and the parameters equal.

2. In the Pyramidal system the axes are rectangular and two of the parameters equal. In this system we shall always suppose a and b equal.

3. In the Rhombohedral system the axes make equal angles with each other, and the parameters are equal.

4. In the Prismatic system the axes are rectangular.

5. In the Oblique-Prismatic system one axis is perpendicular to each of the other two. We shall always suppose the axis OY perpendicular to each of the axes OZ and OX.

6. The Doubly-Oblique-Prismatic system includes all crystals which cannot be referred to either of the preceding systems.

33. The different systems of crystallization are further distinguished by the various kinds of symmetry observable in the distribution of the faces of the crystals belonging to them. For, if a face occur having the symbol $(h\,k\,l)$, it will generally be accompanied by the faces having for their symbols certain arrangements of $\pm h$, $\pm k$, $\pm l$, determined by laws peculiar to each system, and which will be fully explained when we come to describe each system separately.

34. "A form" in crystallography is the figure bounded by a given face and the faces which, by the laws of symmetry of the system of crystallization, are required to coexist with it. A form will be denoted by the symbol of any one of its faces enclosed in braces. Thus, the symbol $\{h\,k\,l\}$ will be used to express the form bounded by the face $(h\,k\,l)$ and its coexistent faces.

The "holohedral forms" of any system are those which possess the highest degree of symmetry of which the system admits. "Hemihedral forms" are those which may be derived from a holohedral form by supposing half of the faces of the latter omitted according to a certain law.

The figure bounded by the faces of any number of forms, is called a "combination" of those forms.

35. The elements of a crystal are the inclinations of the axes YZ, ZX, XY, and the ratios of two of the parameters a, b, c to the third. In the octahedral system, where the axes are rectangular and parameters equal, all the elements are determined. In the pyramidal system, where the axes are rectangular and two of the parameters are equal, the ratio of either of them to the third, is the only variable element. In the rhombohedral system, where the axes make equal angles and the parameters are equal, the angle between any two of the axes is the only variable element. In the prismatic system, where the axes are rectangular, the ratios of two of the parameters to the third, are two variable elements. In the oblique-prismatic system, where one of the axes is at right angles to the other two, the inclination of the two axes which are perpendicular to the third, and the ratios of two of the parameters to the third, are three variable elements. In the doubly-oblique-prismatic system, the angles between the axes, and the ratios of two of the parameters to the third, are all variable.

36. The angle between two faces, the symbols of which are known, may be expressed in terms of the indices of the faces, the inclinations of the axes, and the ratios of the parameters. Therefore, one observed angle between two known faces, in the pyramidal and rhombohedral systems; two observed angles in the prismatic; three in the oblique prismatic, and five in the doubly-oblique-prismatic system, are sufficient to determine the variable elements in the respective systems. In the three latter systems, however, except in particular cases, the above direct method of finding the elements of a crystal is impracticable on account of the high dimensions of the resulting equations. Methods of deducing the elements of a crystal from the requisite number of angles between faces properly selected, adapted to each particular system, will be given in the chapter devoted to that system.

CHAPTER II.

OCTAHEDRAL SYSTEM.

37. In the octahedral system the crystallographic axes are at right angles to each other, and the parameters a, b, c are all equal.

38. The holohedral form $\{h\,k\,l\}$ is bounded by all the faces having for their symbols the different arrangements of $\pm h$, $\pm k$, $\pm l$, taken three at a time. When h, k, l are all different, they afford the forty-eight arrangements contained in the annexed table. When the values of any two of the indices are equal, or when one of them is zero, the number of arrangements will reduce itself to twenty-four. When two of the indices are equal, and the third is zero, the number will be twelve. When the three indices are equal, it will be eight, and when two indices are zero, it will be six.

$h\,k\,l$	$k\,l\,h$	$l\,h\,k$	$l\,k\,h$	$k\,h\,l$	$h\,l\,k$
$h\,\bar{k}\,\bar{l}$	$k\,\bar{l}\,\bar{h}$	$l\,\bar{h}\,\bar{k}$	$l\,\bar{k}\,\bar{h}$	$k\,\bar{h}\,\bar{l}$	$h\,\bar{l}\,\bar{k}$
$\bar{h}\,k\,\bar{l}$	$\bar{h}\,l\,\bar{h}$	$\bar{l}\,h\,\bar{k}$	$\bar{l}\,k\,\bar{h}$	$\bar{k}\,h\,\bar{l}$	$\bar{h}\,l\,\bar{h}$
$\bar{h}\,\bar{k}\,l$	$\bar{k}\,\bar{l}\,h$	$\bar{l}\,\bar{h}\,k$	$\bar{l}\,\bar{k}\,h$	$\bar{k}\,\bar{h}\,l$	$\bar{h}\,\bar{l}\,k$

$\bar{h}\,\bar{k}\,\bar{l}$	$\bar{k}\,\bar{l}\,\bar{h}$	$\bar{l}\,\bar{h}\,\bar{k}$	$\bar{l}\,\bar{k}\,\bar{h}$	$\bar{k}\,\bar{h}\,\bar{l}$	$\bar{h}\,\bar{l}\,\bar{k}$
$\bar{h}\,k\,l$	$\bar{k}\,l\,h$	$\bar{l}\,h\,k$	$\bar{l}\,k\,h$	$\bar{k}\,h\,l$	$\bar{h}\,l\,k$
$h\,\bar{k}\,l$	$k\,\bar{l}\,h$	$l\,\bar{h}\,k$	$l\,\bar{k}\,h$	$k\,\bar{h}\,l$	$h\,\bar{l}\,k$
$h\,k\,\bar{l}$	$k\,l\,\bar{h}$	$l\,h\,\bar{k}$	$l\,k\,\bar{h}$	$k\,h\,\bar{l}$	$h\,l\,\bar{k}$

If we suppose h to be the greatest, and l the least of the three unequal indices h, k, l, fig. 9, will represent the distribution of the poles of the form $\{h\,k\,l\}$ on the surface of the sphere of projection. Fig. 10. exhibits the poles of the forms obtained by substituting zero for one of the indices, or by making two of them equal. Both figures shew the poles of the forms $\{1\,0\,0\}$, $\{1\,1\,1\}$ and $\{0\,1\,1\}$.

39. The form bounded either by all the faces of $\{h\,k\,l\}$ which have an odd number of positive indices, or by all the faces of $\{h\,k\,l\}$ which have an odd number of negative indices, is said to be hemihedral with inclined faces, and will be denoted by the symbol $\kappa\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one of its faces. The hemihedral form bounded by the faces which have an odd number of positive indices, is said to be direct. The form bounded by the faces which have an odd number of negative indices, is said to be inverse. The upper and lower halves of the table in (38), contain the symbols of the faces of the direct and inverse forms respectively.

If the surface of the sphere of projection be divided into eight triangles by zone-circles through every two of the poles of the form $\{1\,0\,0\}$, the poles of the direct hemihedral form will be found in four alternate triangles, one of which contains the poles of $(1\,1\,1)$; and the poles of the inverse hemihedral form will be found in the remaining four alternate triangles

40. The form bounded either by all the faces of $\{h\,k\,l\}$ the indices of which stand in the order $h\,k\,l\,h\,k$, or by all the faces of $\{h\,k\,l\}$ the indices of which stand in the order $l\,k\,h\,l\,k$, is said to be hemihedral with parallel faces, and will be denoted by the symbol $\pi\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one of its faces. The form is said to be direct or inverse according as the numerical values of the indices of one of its faces are in ascending or descending order. The symbols of the faces of the direct and inverse forms are

contained respectively in the right and left halves of the table in (38).

If the surface of the sphere of projection be divided into twenty-four triangles by zone-circles passing through every two of the poles of the form $\{1\ 1\ 1\}$, the poles of the direct hemihedral form will be found in twelve alternate triangles one of which is $(1\ 1\ 1)$ $(0\ 1\ 0)$ $(1\ 1\ \bar{1})$, and the poles of the inverse form will be found in the remaining twelve alternate triangles.

41. Any number of holohedral forms may occur in combination with each other, and with any hemihedral forms with inclined faces, or with any hemihedral forms with parallel faces. It is said that hemihedral forms with inclined faces have never been observed in combination with hemihedral forms with parallel faces.

42. To find the position of the pole of any face.

Let the axes of the crystal meet the surface of the sphere of projection in X, Y, Z, (fig. 11); and let P be the pole of the face $(h\ k\ l)$. The axes are rectangular; therefore YZ, ZX, XY are quadrants, therefore $\cos YZ = 0$, $\cos ZX = 0$, $\cos XY = 0$, therefore X, Y, Z are the poles of the faces $(1\ 0\ 0)$, $(0\ 1\ 0)$, $(0\ 0\ 1)$ respectively. The quadrantal triangles PYX, PYZ give

$$(\cos PY)^2 = (\sin PX)^2 (\cos PXY)^2,$$

$$(\cos PZ)^2 = (\sin PX)^2 (\cos PXZ)^2.$$

Add, observing that $(\cos PXY)^2 + (\cos PXZ)^2 = 1$, and that $(\cos PX)^2 + (\sin PX)^2 = 1$, and we have

$$(\cos PX)^2 + (\cos PY)^2 + (\cos PZ)^2 = 1.$$

The parameters are equal, therefore (8),

$$\frac{1}{h} \cos PX = \frac{1}{k} \cos PY = \frac{1}{l} \cos PZ,$$

$$\therefore \quad (\cos PX)^2 = \frac{h^2}{h^2+k^2+l^2},$$

$$(\cos PY)^2 = \frac{k^2}{h^2+k^2+l^2},$$

$$(\cos PZ)^2 = \frac{l^2}{h^2+k^2+l^2}.$$

43. To find the distance between the poles of any two faces.

Let P (fig. 11) be the pole of $(h\,k\,l)$, Q the pole of $(p\,q\,r)$.

$$\cos PXQ = \cos PXY \cos QXY + \sin PXY \sin QXY,$$
$$= \cos PXY \cos QXY + \cos PXZ \cos QXZ.$$

Substituting this value of $\cos PXQ$ in the equation

$$\cos PQ = \cos PX \cos QX + \sin PX \sin QX \cos PXQ,$$

and observing that

$$\sin PX \cos PXY = \cos PY, \quad \sin QX \cos QXY = \cos QY,$$

$$\sin PX \cos PXZ = \cos PZ, \quad \sin QX \cos QXZ = \cos QZ,$$

we get

$$\cos PQ = \cos PX \cos QX + \cos PY \cos QY + \cos PZ \cos QZ.$$

But (42)

$$(\cos PX)^2 = \frac{h^2}{h^2+k^2+l^2}, \quad (\cos QX)^2 = \frac{p^2}{p^2+q^2+r^2},$$

$$(\cos PY)^2 = \frac{k^2}{h^2+k^2+l^2}, \quad (\cos QY)^2 = \frac{q^2}{p^2+q^2+r^2},$$

$$(\cos PZ)^2 = \frac{l^2}{h^2+k^2+l^2}, \quad (\cos QZ)^2 = \frac{r^2}{p^2+q^2+r^2}.$$

$$\therefore \quad \cos PQ = \frac{hp+kq+lr}{\sqrt{(h^2+k^2+l^2)}\sqrt{(p^2+q^2+r^2)}}.$$

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44. The three quadrantal triangles YPZ, ZPX, XPY, give

$$\cos PX = \sin PY \cos PYX = \sin PZ \cos PZX,$$

$$\cos PY = \sin PZ \cos PZY = \sin PX \cos PXY,$$

$$\cos PZ = \sin PX \cos PXZ = \sin PY \cos PYZ.$$

Whence

$$\tan PXY = \frac{l}{k}, \quad \tan PYZ = \frac{h}{l}, \quad \tan PZX = \frac{k}{h}.$$

45. Let O be the pole of the face $(1\,1\,1)$, $\therefore$ (42), $(\cos OX)^2 = \frac{1}{3}$, $(\cos OY)^2 = \frac{1}{3}$, $(\cos OZ)^2 = \frac{1}{3}$; whence (43),

$$(\cos\ PO)^2 = \frac{1}{3}\,\frac{(h+k+l)^2}{h^2+k^2+l^2}.$$

$OX = OY = OZ$, and YZ, ZX, XY are quadrants. Hence, the angles YOZ, ZOX, XOY are each equal to 120°, and OX, OY, OZ bisect the right angles YXZ, ZYX, XZY. From the triangle POX we have

$$\cot PX \sin XO = \cos XO \cos OXP + \sin OXP \cot POX.$$

If we write Y, Z successively in the place of X, and subtitute for the trigonometrical ratios their values in terms of h, k, l, we obtain

$$\tan POX = \sqrt{3}\,\frac{k-l}{2h-k-l},$$

$$\tan POY = \sqrt{3}\,\frac{l-h}{2k-l-h},$$

$$\tan POZ = \sqrt{3}\,\frac{h-k}{2l-h-k}.$$

46. It appears from the form of the expressions in (43), that the distance between the poles of the faces $(h\,k\,l)$, $(p\,q\,r)$ is equal to the distance between the pole of any face of the form $\{h\,k\,l\}$, and the pole of any face

of the form $\{p\,q\,r\}$, in the symbols of which the order and signs of h, k, l are the same as the order and signs of p, q, r.

47. It appears from the expressions in (42), that if the symbols of two poles of the form $\{h\,k\,l\}$ differ only in the sign of h, the two poles will be equidistant from (0 1 0) and also from (0 0 1), therefore the arc joining the two poles will be bisected at right angles by the zone-circle [0 1 0, 0 0 1]. Hence the poles of $\{h\,k\,l\}$ are symmetrically arranged with respect to the zone-circle [0 1 0, 0 0 1]. In like manner it may be shewn, that the poles of $\{h\,k\,l\}$ are symmetrically situated with respect to any one of the three zone-circles that can be drawn through every two of the poles of $\{1\,0\,0\}$.

48. It appears from (43), that if the symbols of two poles of the form $\{h\,k\,l\}$ differ only in the arrangement of the 2nd and 3rd indices, the poles will be equidistant from ($\bar{1}$ 1 1) and also from (1 1 1), therefore the arc joining the two poles will be bisected at right angles by the zone-circle [$\bar{1}$ 1 1, 1 1 1]. Hence the poles of $\{h\,k\,l\}$ are symmetrically arranged with respect to the zone-circle [$\bar{1}$ 1 1, 1 1 1]. In like manner it may be shewn, that the poles of $\{h\,k\,l\}$ are symmetrically arranged with respect to any one of the six zone-circles that can be drawn through every two of the poles of $\{1\,1\,1\}$.

49. If zone-circles be drawn through every two of the poles of $\{1\,0\,0\}$, and through every two of the poles of $\{1\,1\,1\}$, they will divide the surface of the sphere of projection into forty-eight right-angled triangles. The poles of $\{h\,k\,l\}$ are symmetrically arranged with respect to any side of any one of the triangles. Hence the arrangement of the poles will be symmetrical in any two adjacent triangles, and similar in any two alternate triangles.

50. If zone-circles be drawn through every two of the poles of $\{1\,0\,0\}$, and through every two of the poles of

$\{1\,1\,1\}$, the zone-circles of one set bisect symmetrically the triangles formed by the zone-circles of the second set. Hence, the poles of $\kappa\{h\,k\,l\}$ are symmetrically arranged with respect to the zone-circles drawn through every two of the poles of $\{1\,1\,1\}$; and the poles of $\pi\{h\,k\,l\}$ are symmetrically arranged with respect to the zone-circles drawn through every two of the poles of $\{1\,0\,0\}$.

51. If we examine the situations of the poles of two hemihedral forms, either with inclined or parallel faces, derived from the same holohedral form, one of them being direct and the other inverse, we shall find that the two forms are identical in all respects, position excepted, and that one of them may be brought into the position of the other by making it revolve through a right angle round any one of its crystallographic axes. In like manner, the combinations of a holohedral form with a direct and inverse hemihedral form differ only in position. But in the combinations of any two hemihedral forms with inclined faces, or of any two hemihedral forms with parallel faces, with each other, the poles of the two forms lie in the same or in different triangles according as they are of the same or different denominations. Hence a combination of two direct, or of two inverse hemihedral forms, is essentially different from the combination of the same hemihedral forms, when one of them is direct and the other inverse.

52. If the distance between the poles of any two faces of either of the forms $\{h\,k\,0\}$, $\{h\,k\,k\}$ be given, and we express the cosine of the given distance in terms of the indices of the faces, we obtain an equation from which the ratio of the indices may be deduced.

53. If the distances between the pole of any face of the form $\{h\,k\,l\}$, and the poles of each of two other faces of the same form be given, and we express the cosines of the given distances in terms of the indices of the faces, we shall obtain two equations from which the ratios of the indices may be readily found.

54. To determine the figure and angles of the form $\{h\,k\,l\}$, when h, k, l take particular values.

The angle between normals to any two faces, which is measured by the angular distance between their poles, is found by substituting the indices of the faces for h, k, l, p, q, r in the expressions of (43). The letter placed upon the edge resulting from the intersection of any two faces, in the figures which accompany the description of each particular form, will be used to denote the angle between normals to the two faces. The same letter will be placed upon all the edges at which equal angles are formed by the intersecting faces. The relative positions of the poles of the different forms are shewn in figs. 9, 10, and in fig. 37, which is the gnomonic projection of one of the octants into which the surface of the sphere of projection is divided by the zone-circles through every two of the poles of the form $\{1\,0\,0\}$. The number of faces of each holohedral form has been already determined in (38).

55. The form $\{1\,0\,0\}$ (fig. 12) has six faces, and is called a cube.

$$\cos F = 1, \quad \therefore\ F = 90^0.$$

Hence, the faces of the form $\{1\,0\,0\}$ are parallel to those of a cube.

56. The form $\{1\,1\,1\}$ (fig. 13) has eight faces, and is called an octahedron.

$$\cos D = \tfrac{1}{3}, \quad \therefore\ D = 70^0.31',7.$$

Hence, the faces of the form $\{1\,1\,1\}$ are parallel to those of a regular octahedron.

The cosine of the angle between normals to any face of the octahedron and either of the adjacent faces of the cube is $\frac{1}{3}\sqrt{3}$. Hence, a normal to any face of the octahedron makes an angle of $54^0.44',15$ with the normals to each of the adjacent faces of the cube, and an angle of $125^0.15',85$ with the normals to each of the three opposite faces of the cube.

57. In the hemioctahedron with inclined faces $\kappa\{1\,1\,1\}$ (fig. 14),

$$\cos T = -\tfrac{1}{3}, \quad \therefore\ T = 109^0.28',3.$$

Hence, the hemihedral form $\kappa\,\{1\,1\,1\}$ is a regular tetrahedron.

58. The form $\{0\,1\,1\}$ (fig. 15) has twelve faces, and is called a dodecahedron.

$$\cos G = \tfrac{1}{2}, \quad \therefore\ G = 60^0.$$

If normals to any two alternate faces, meeting at their acute angles, make an angle D with each other, $\cos D = 0$, $\therefore\ D = 90^0$.

The cosine of the angle between normals to any face of the dodecahedron and either of the adjacent faces of the cube is $\frac{1}{2}\sqrt{2}$. The cosine of the angle between normals to any face of the dodecahedron and either of the faces of the cube, not parallel to the two adjacent faces, is 0. Hence, a normal to any face of the dodecahedron makes an angle of 45^0 with the normals to each of the two adjacent faces of the cube; an angle of 135^0 with the normals to each of the two opposite faces, and an angle of 90^0 with the normals to each of the two remaining faces.

The cosine of the angle between normals to any face of the dodecahedron and either of the adjacent faces of the octahedron is $\frac{1}{3}\sqrt{6}$. The cosine of the angle between normals to any face of the dodecahedron and either of the faces of the octahedron, not parallel to the adjacent faces, is 0. Hence, a normal to any face of the dodecahedron makes an angle of $35^0.15',85$ with the normals to each of the two adjacent faces of the octahedron; an angle of $144^0.44',15$ with the normals to each of the two opposite faces, and an angle of 90^0 with the normals to each of the four remaining faces.

59. The form $\{h\,k\,0\}$ (fig. 16) has twenty-four faces and is called a tetrakishexahedron.

$$\cos F = \frac{2\,hk}{h^2 + k^2}, \quad \cos G = \frac{h^2}{h^2 + k^2}.$$

In $\{2\,1\,0\}$, $\cos F = \frac{4}{5}$, $\cos G = \frac{4}{5}$,

$$\therefore\ F = 36^0.52',2, \quad G = 36^0.52',2.$$

In $\{3\,1\,0\}$, $\cos F = \frac{6}{10}$, $\cos G = \frac{9}{10}$,

$$\therefore\ F = 53^0.7',8, \quad G = 25^0.50',5.$$

In $\{3\,2\,0\}$, $\cos F = \frac{12}{13}$, $\cos G = \frac{9}{13}$,

$$\therefore\ F = 22^0.37',2, \quad G = 46^0.11',2.$$

In $\{5\,2\,0\}$, $\cos F = \frac{20}{29}$, $\cos G = \frac{25}{29}$,

$$\therefore\ F = 46^0.23',8, \quad G = 30^0.27'.$$

The tangent of the angle between normals to any face of $\{h\,k\,0\}$ and the nearest face of $\{1\,0\,0\}$ is $k \div h$.

60. In the hemitetrakishexahedron with parallel faces $\pi\{0\,k\,h\}$ (fig. 17), h being greater than k,

$$\cos D = \frac{h^2 - k^2}{h^2 + k^2}, \quad \cos U = \frac{hk}{h^2 + k^2}.$$

In $\pi\{0\,1\,2\}$, $\cos D = \frac{3}{5}$, $\cos U = \frac{2}{5}$,

$$\therefore\ D = 53^0.7',8, \quad U = 66^0.25',3.$$

In $\pi\{0\,2\,3\}$, $\cos D = \frac{5}{13}$, $\cos U = \frac{6}{13}$,

$$\therefore\ D = 67^0.22',1, \quad U = 62^0.30',8.$$

In $\pi\{0\,3\,4\}$, $\cos D = \frac{7}{25}$, $\cos U = \frac{12}{25}$,

$$\therefore\ D = 73^0.44,4, \quad U = 61^0.18',9.$$

61. The form $\{h\,k\,k\}$ (fig. 18), h being greater than k, has twenty-four faces, and is called an icositessarahedron.

$$\cos D = \frac{h^2}{h^2 + 2k^2}, \quad \cos F = \frac{2hk + k^2}{h^2 + 2k^2}.$$

In $\{2\,1\,1\}$, $\cos D = \frac{4}{6}$, $\cos F = \frac{5}{6}$,

$$\therefore\ D = 48^0.11,5, \quad F = 33^0.33',4.$$

In $\{3\,1\,1\}$, $\cos D = \frac{9}{11}$, $\cos F = \frac{7}{11}$,

$$\therefore\ D = 35^0.5',8, \quad F = 50^0.28',7.$$

62. In the hemiicositessarahedron with inclined faces $\kappa\{h\,k\,k\}$ (fig. 19),

$$\cos F = \frac{2hk + k^2}{h^2 + 2k^2}, \quad \cos T = \frac{h^2 - 2k^2}{h^2 + 2k^2}.$$

In $\kappa\{2\,1\,1\}$, $\cos F = \frac{5}{6}$, $\cos T = \frac{2}{6}$,

$$\therefore\ F = 33^0.\,33',4, \quad T = 70^0.\,31',7.$$

In $\kappa\{3\,1\,1\}$, $\cos F = \frac{7}{11}$, $\cos T = \frac{7}{11}$,

$$\therefore\ F = 50^0.\,28',7, \quad T = 50^0.\,28',7.$$

63. The form $\{h\,h\,k\}$ (fig. 20), h being greater than k, has twenty-four faces, and is called a triakisoctahedron.

$$\cos D = \frac{2h^2 - k^2}{2h^2 + k^2}, \quad \cos G = \frac{h^2 + 2hk}{2h^2 + k^2}.$$

In $\{2\,2\,1\}$, $\cos G = \frac{8}{9}$, $\cos D = \frac{7}{9}$,

$$\therefore\ G = 27^0.16', \quad D = 38^0.56,5.$$

In $\{3\,3\,1\}$, $\cos G = \dfrac{15}{19}$, $\cos D = \dfrac{17}{19}$,

$$\therefore\ G = 37^0.\,51',8,\ D = 26^0.\,31',5.$$

64. In the hemitriakisoctahedron with inclined faces $\kappa\{h\,h\,k\}$ (fig. 21),

$$\cos G = \frac{h^2 + 2hk}{2h^2 + k^2},\quad \cos T = \frac{h^2 - 2hk}{2h^2 + k^2}.$$

In $\kappa\{2\,2\,1\}$, $\cos G = \dfrac{8}{9}$, $\cos T = 0$,

$$\therefore\ G = 27^0.16',\ T = 90^0.$$

65. The form $\{h\,k\,l\}$ (fig. 22) has forty-eight faces, and is called a hexakisoctahedron.

$$\cos D = \frac{h^2 + k^2 - l^2}{h^2 + k^2 + l^2},\ \cos F = \frac{2hk + l^2}{h^2 + k^2 + l^2},\ \cos G = \frac{h^2 + 2kl}{h^2 + k^2 + l^2}.$$

In $\{3\,2\,1\}$, $\cos D = \dfrac{12}{14}$, $\cos F = \dfrac{13}{14}$, $\cos G = \dfrac{13}{14}$,

$$\therefore\ D = 31^0.0',2,\ F = 21^0.47',2,\ G = 21^0.47',2.$$

In $\{4\,3\,1\}$, $\cos D = \dfrac{24}{26}$, $\cos F = \dfrac{25}{26}$, $\cos G = \dfrac{22}{26}$,

$$\therefore\ D = 22^0.\,37',2,\ F = 15^0.\,56',5,\ G = 32^0.\,12',2.$$

In $\{4\,2\,1\}$, $\cos D = \dfrac{19}{21}$, $\cos F = \dfrac{17}{21}$, $\cos G = \dfrac{20}{21}$,

$$\therefore\ D = 25^0.\,12',5,\ F = 35^0.\,57',\ G = 17^0.\,45,1.$$

In $\{7\,3\,1\}$, $\cos D = \dfrac{57}{59}$, $\cos F = \dfrac{43}{59}$, $\cos G = \dfrac{55}{59}$,

$$\therefore\ D = 14^0.\,57',7,\ F = 43^0.\,12',8,\ G = 21^0.\,13',2.$$

In $\{11\,5\,3\}$, $\cos D = \dfrac{137}{155}$, $\cos F = \dfrac{151}{155}$, $\cos G = \dfrac{119}{155}$,

$\therefore$ $D = 27^0.\,53',2$, $F = 39^0.\,50',9$, $G = 13^0.\,2',7$.

66. In the hemihexakisoctahedron with inclined faces $\kappa\{h\,k\,l\}$ (fig. 23),

$$\cos F = \frac{2hk + l^2}{h^2 + k^2 + l^2}, \quad \cos G = \frac{h^2 + 2kl}{h^2 + k^2 + l^2}, \quad \cos T = \frac{h^2 - 2kl}{h^2 + k^2 + l^2}.$$

In $\kappa\{3\,2\,1\}$, $\cos F = \dfrac{13}{14}$, $\cos G = \dfrac{13}{14}$, $\cos T = \dfrac{5}{14}$,

$\therefore$ $F = 21^0.\,47',2$, $G = 21^0.\,47',2$, $T = 69^0.\,4',5$.

In $\kappa\{5\,3\,1\}$, $\cos F = \dfrac{31}{35}$, $\cos G = \dfrac{31}{35}$, $\cos T = \dfrac{19}{35}$,

$\therefore$ $F = 27^0.\,39',7$, $G = 27^0.\,39',7$, $T = 57^0.\,7',3$.

67. In the hemihexakisoctahedron with parallel faces, $\pi\{l\,k\,h\}$ (fig. 24),

$$\cos D = \frac{h^2 + k^2 - l^2}{h^2 + k^2 + l^2}, \quad \cos W = \frac{h^2 - k^2 + l^2}{h^2 + k^2 + l^2}, \quad \cos U = \frac{kl + lh + hk}{h^2 + k^2 + l^2}.$$

In $\pi\{1\,2\,3\}$, $\cos W = \dfrac{6}{14}$, $\cos D = \dfrac{12}{14}$, $\cos U = \dfrac{11}{14}$,

$\therefore$ $W = 64^0.\,37',3$, $D = 39^0.\,0',3$, $U = 38^0.\,12',8$.

In $\pi\{1\,2\,4\}$, $\cos W = \dfrac{13}{21}$, $\cos D = \dfrac{19}{21}$, $\cos U = \dfrac{14}{21}$,

$\therefore$ $W = 51^0.\,45',3$, $D = 25^0.\,12',7$, $U = 48^0.\,11',3$.

In $\pi\{1\,3\,5\}$, $\cos W = \dfrac{17}{35}$, $\cos D = \dfrac{33}{35}$, $\cos U = \dfrac{23}{35}$,

$\therefore$ $W = 29^0.\,3,5$, $D = 19^0.\,27',8$, $U = 48^0.\,55'$.

{hkk} $\dfrac{k}{h} = \dfrac{\tan AP}{\sqrt{2}}$ {hko} $\dfrac{h}{k} = \cot AQ.$

68. Of the preceding forms, those which have the simplest indices, the cube {1 0 0}, octahedron {1 1 1} and dodecahedron {0 1 1} occur much more frequently than the others. It does not appear that any cleavages have been observed except parallel to the faces of one or more of the three forms {1 0 0}, {1 1 1}, {0 1 1}.

Table of the distances between the poles of the commonest forms, and the nearest poles of the forms {1 0 0}, {0 1 1}, {1 1 1}.

	100	010	001	011	101	110	111
	° ′	° ′	° ′	° ′	° ′	° ′	° ′
210	26.34	63.26	90. 0	71.34	50.46	18.26	39.44
310	18.26	71.34	90. 0	77. 5	47.52	26.34	43. 5
320	33.41	56.19	90. 0	66.54	53.58	11.19	36.49
410	14. 2	75.58	90. 0	80. 7	46.41	30.58	45.34
430	36.52	53. 8	90. 0	64.54	55.33	8. 8	36. 4
520	21.48	68.12	90. 0	74.47	48.58	23.12	41.22
540	38.40	51.20	90. 0	63.47	56.29	6.20	35.45
211	35.16	65.54	65.54	54.44	30. 0	30. 0	19.28
311	25.14	72.27	72.27	64.46	31.29	31.29	29.30
122	70.31	48.11	48.11	19.28	45. 0	45. 0	15.48
133	76.44	46.30	46.30	13.16	49.33	49.33	22. 0
321	36.42	57.41	74.30	55.28	40.54	19. 6	22.13
421	29.12	64. 7	77.24	62.25	39.31	22.13	28. 8
431	38.20	53.58	78.41	56.19	46. 6	13.54	25. 4
531	32.19	59.32	80.16	61.26	44.11	17. 1	28.35
731	24.18	67. 1	82.31	68.24	42.34	22.59	34.14
511	15.47½						38.56½
722	22.0	74.38					32.44
740	29.45					15.15	38.1½
720	15.57					29.3	44.28
522	29.30	69.37½				30.30	30.30
411	19.28	76.22		70.32		33.33½	
233	64.45½			25.14½			10.1½
344	62.6			27.56			7.20
477	68.0	49.2		22.0			13.16
~~255~~				~~19.28½~~			~~15.47½~~
255	74.12½			15.47½			19.28½

510

Examples.

69. In a crystal of Tin-white Cobalt Pyrites, (fig. 25.) the normals to the faces a, a', &c. make right angles with each other, and a normal to any of the faces d, d', &c. makes an angle of 45^0 with a normal to either of the adjacent faces a, a', &c. Hence, (55), (58), a, a', &c. are the faces of the cube $\{1\,0\,0\}$, and d, d', &c. are the faces of the dodecahedron $\{0\,1\,1\}$. Let symbols of the faces be a $(1\,0\,0)$, a' $(0\,1\,0)$, a'' $(0\,0\,1)$. Therefore d $(0\,1\,1)$, d' $(1\,0\,1)$, d'' $(1\,1\,0)$. o is in the zone ad, and also in the zone $a'd'$. The symbols of these zones are ad $[0\,1\,\bar{1}]$, $a'd'$ $[\bar{1}\,1\,0]$ (14). Therefore (15) o is $(1\,1\,1)$, therefore o is a face of the octahedron $\{1\,1\,1\}$. Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{0\,1\,1\}$, $\{1\,1\,1\}$.

70. In a crystal of Fluor (fig. 26), normals to any two adjacent faces o make with each other an angle of $70^0.\,31'\frac{1}{2}$, therefore (56) o, o', &c. are the faces of the octahedron $\{1\,1\,1\}$. A normal to any face f makes an angle of 22^0 with a normal to the adjacent face o. The edges in which any two adjacent faces f meet each other and the adjacent faces o are parallel, therefore the faces f are in the same zone with two adjacent faces o. Let o be $(1\,1\,1)$, o' $(\bar{1}\,1\,1)$, therefore the zone oo' will be $[0\,1\,\bar{1}]$, therefore (21) the symbol of f will be $(k\,h\,h)$, and that of f $(\bar{k}\,h\,h)$. If P, Q be the poles of f, f', $PQ = 70^0.31'\frac{1}{2} - 2.22^0 = 26^0.31'\frac{1}{2}$, and

$$\frac{2h^2 - k^2}{2h^2 + k^2} = \cos PQ = \cos 26^0.31'\tfrac{1}{2} = \frac{17}{19}, \quad \therefore \frac{h^2}{k^2} = 9, \quad \frac{h}{k} = 3.$$

Therefore f is a face of the triakisoctahedron $\{3\,3\,1\}$. Hence the crystal is a combination of the forms $\{1\,1\,1\}$, $\{3\,3\,1\}$. It is cleavable parallel to the faces of $\{1\,1\,1\}$.

71. In a crystal of Fluor (fig. 27), the angle between normals to any two adjacent faces n are alternately $35^0.57'$ and $17^0.45'$, the angle between the normals to the faces

that meet in a long edge being the greater of the two. Let X, O be the poles of $(1\,0\,0)$, $(1\,1\,1)$; $\{h\,k\,l\}$ the symbol of the form bounded by the faces n; P the pole of $(h\,k\,l)$, Q the pole of $(k\,h\,l)$ and R the pole of $(h\,l\,k)$. Then $PQ = 35^0.57'$, $PR = 17^0.45'$.

$$\cos PQ = \frac{2hk + l^2}{h^2 + k^2 + l^2}, \quad \cos PR = \frac{h^2 + 2kl}{h^2 + k^2 + l^2}.$$

$$\cos PQ = \frac{17}{21}, \ \cos PR = \frac{20}{21}; \ \therefore \frac{(h-k)^2}{h^2+k^2+l^2} = \frac{4}{21}, \ \frac{(k-l)^2}{h^2+k^2+l^2} = \frac{1}{21}.$$

$$\text{Whence } \frac{(h-l)^2}{h^2+k^2+l^2} = \frac{9}{21}, \ \frac{k^2+2hl}{h^2+k^2+l^2} = \frac{12}{21}, \ \frac{(h+k+l)^2}{h^2+k^2+l^2} = \frac{49}{21}.$$

$$\therefore \ h - k = 2(k - l), \ h + k + l = 7(k - l).$$

Whence $k = 2l$, $h = 4l$, $\therefore l = 1$, $k = 2$, $h = 4$. Hence n, n', &c. are faces of the form $\{4\,2\,1\}$.

72. In a crystal of Boracite (fig. 28), normals to any two adjacent faces a makes right angles with each other, they are therefore the faces of a cube. Let their symbols be a $(1\,0\,0)$, a' $(0\,1\,0)$, a'' $(0\,0\,1)$. A normal to d'' makes an angle of 45^0 with a normal to either of the faces a, a', therefore (58), d'' is $(1\,1\,0)$. Similarly d' is $(1\,0\,1)$, and d is $(0\,1\,1)$. o is common to the zones ad, $a'd'$, therefore (56) o is $(1\,1\,1)$. The number of faces o is four. But the holohedral form $\{1\,1\,1\}$ has eight faces. Therefore o, o, &c. are the faces of the hemioctahedron $\kappa\{1\,1\,1\}$. Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{1\,1\,0\}$, $\kappa\{1\,1\,1\}$.

73. In a crystal of Magnetic Iron Oxide, (fig. 29), normals to any two adjacent faces o make with each other an angle of $70^0.31'\frac{1}{2}$. Therefore o, o', &c. are the faces of the octahedron $\{1\,1\,1\}$. Let the symbols of the faces be o $(1\,1\,1)$, o' $(\bar{1}\,1\,1)$, o'' $(1\,1\,\bar{1})$, o''' $(\bar{1}\,1\,\bar{1})$. A normal to any face d makes angles of $35^0.16'$ with normals to any two adjacent faces o, therefore d, d', &c. are the faces of the

dodecahedron $\{0\,1\,1\}$, e is common to the zones oo''', do'', the symbols of which are oo''' $[\bar{1}\,0\,1]$, do'' $[2\,1\,1]$, therefore (15), e is $(1\,3\,1)$; therefore e, e', &c. are faces of the icositetrahedron $\{3\,1\,1\}$. Hence the crystal is a combination of of the forms $\{1\,1\,1\}$, $\{0\,1\,1\}$, $\{3\,1\,1\}$.

74. In a crystal of Garnet (fig. 30), the normals to any two adjacent faces d make with each other an angle of 60^0, therefore d, d', &c. are the faces of the dodecahedron $\{0\,1\,1\}$. Let their symbols be d $(0\,1\,1)$, d' $(1\,0\,1)$, d'' $(1\,1\,0)$. e' is in the zone dd', and makes equal angles with d, d'. The zone-circle $[1\,1\,1,\ 1\,\bar{1}\,1]$ bisects the arc dd' (48), therefore it contains the pole of e'. Therefore e' is $(1\,2\,1)$. Similarly $e_{,}$ is $(1\,2\,\bar{1})$. s is in the zone $e'e_{,}$; it is also in the zone dd', therefore (15), s is $(1\,2\,3)$. Hence the crystal is a combination of the forms $\{0\,1\,1\}$, $\{2\,1\,1\}$, $\{1\,2\,3\}$. It is cleavable parallel to the faces of $\{0\,1\,1\}$.

75. In a crystal of Silver-white Cobalt from Tunaberg (fig. 31), normals to any two adjacent faces a make an angle of 90^0 with each other, they are therefore the faces of the cube. Let their symbols be a $(1\,0\,0)$, a' $(0\,1\,0)$, a'' $(0\,0\,1)$. The angle between a normal to any face o and a normal to any of the adjacent faces a is $54^0.44$, therefore (56) o is $(1\,1\,1)$. The edges which d'' make with a and a' are parallel, therefore d'' is in the same zone with a, a'; therefore the symbol of d'' will be $(h\,k\,0)$. It is found that normals to a', d'' make an angle of $26^0.34'$. Therefore (42)

$$\frac{k^2}{h^2+k^2} = (\cos 26^0.34')^2, \quad \therefore \frac{h}{k} = \tan 26^0.34' = \tfrac{1}{2}, \quad \therefore d'' \text{ is } (1\,2\,0).$$

The number of faces d is twelve, the number in the holohedral form $\{0\,1\,2\}$ being twenty-four. Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{1\,1\,1\}$, $\pi\{0\,1\,2\}$. The crystal is cleavable parallel to the faces of $\{1\,0\,0\}$.

76. In a crystal of Silver-white Cobalt (fig. 32), the symbols of p, a, e, k are p $(1\,0\,0)$, a $(1\,1\,1)$, e $(1\,2\,0)$, k $(1\,4\,0)$.

i is in the zone ac, and normals to a, i make, according to Phillips, an angle of $16^0.33'$. Let $(h\,k\,l)$ be the symbol of i. The symbol of the zone ac is $[\bar{2}\,1\,1]$, therefore (21) $2h = k + l$. Also (45) $\frac{1}{3}\frac{(h+k+l)}{h^2+k^2+l^2} = (\cos 16^0.33')^2$. Whence $\frac{k}{l} = \frac{15}{7}$, very nearly, therefore $l = 7$, $k = 15$, $h = 11$. The number of faces e, adjacent to a, is three, therefore e is a face of the form $\pi\{7\ 11\ 15\}$.

77. In a crystal of Yellow Iron Pyrites (fig. 33), a is a face of the cube, o a face of the octahedron. Normals to s, a make with each other an angle of $57^0.41$, and normals to s, o make with each other an angle of $22^0.13'$. Let the symbols of the faces be a (1 0 0), o (1 1 1), s $(h\,k\,l)$; and let A, O, S be the poles of a, o, s. Then $SA = 57^0.41'$, $SO = 22^0.13'$.

$$\therefore (43)\ \frac{h^2}{h^2+k^2+l^2} = (\cos SA)^2 = \frac{9}{14},\ (45)\ \frac{(h+k+l)^2}{3(h^2+k^2+l^2)} = (\cos SO)^2 = \frac{6}{7}.$$

Whence $2k = 3h$, $2l = h$, $\therefore l = 1$, $h = 2$, $k = 3$, $\therefore s$ is (2 3 1). In the holohedral form $\{1\,2\,3\}$, and in the hemihedral form with inclined faces, there are six faces on one angle of the cube. In the present case there are only three. Hence s, s', s'' are faces of the direct hemihexakisoctahedron with parallel faces $\pi\{1\,2\,3\}$. The crystal is cleavable parallel to the faces of the forms $\{1\,0\,0\}$, $\{1\,1\,1\}$.

78. In a crystal of Yellow Iron Pyrites (fig. 34), p, p', p'' are faces of the cube, d a face of the octahedron $p\,y''\,e''\,p'$, $p''\,s\,f\,o'\,s''\,e''$, $p''\,f''\,y''$, $p\,o\,d$, $d\,f\,e$ are zones; and the normals to $p'\,e''$ make an angle of $26^0.34'$. Let the symbols of p, p', p'' be (1 0 0), (0 1 0), (0 0 1), then d will be (1 1 1). e'' is in the zone $p\,p'$, therefore its symbol will be of the form $(h\,k\,0)$; therefore if Y, P be the poles of p', e'', $(\cos PY)^2 = \frac{k^2}{h^2+k^2}$,

$\therefore \frac{h}{k} = \tan PY = \tan 26^0.34' = \frac{1}{2}$, $\therefore e''$ is (1 2 0). In like manner e' is (2 0 1), and e is (0 1 2). o is common to the

zones pd, $p'e'$, therefore o is $(2\,1\,1)$. s is common to the zones pe, $p''e''$, therefore s is $(1\,2\,4)$. f is common to the zones de, $p''e''$, therefore f is $(1\,2\,3)$. y is common to the zones pf, $p'p''$, therefore y is $(0\,2\,3)$. There is only one face e, and one face y, between every two adjacent faces of the cube; and only three faces f, and three faces s, round each of the faces d; therefore e, q, f, s belong to hemihedral forms with parallel faces. Hence, the crystal is a combination of the forms $\{1\,1\,1\}$, $\{1\,0\,0\}$, $\{2\,1\,1\}$, $\pi\{0\,1\,2\}$, $\pi\{0\,2\,3\}$, $\pi\{1\,2\,3\}$, $\pi\{1\,2\,4\}$.

79. In a crystal of Fahlerz (fig. 35), f, f', f'' are the faces of the cube. Each of the faces d, d', &c. is in the same zone with the two adjacent faces a, and makes equal angles with them, therefore d, d', &c. are faces of the dodecahedron. Let f, f', f'' be $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$ respectively, therefore d, d', &c. will be d $(0\,1\,1)$, d' $(1\,0\,1)$, d'' $(1\,1\,0)$, $d_{,}$ $(1\,0\,\bar{1})$. p is common to the zones $f'd_{,}$, $f''d'$, therefore p is $(1\,1\,\bar{1})$. r'' is common to the zones dd', $f''d''$, therefore r'' is $(1\,1\,2)$. In like manner we have r' $(1\,2\,1)$, $r_{,}$ $(2\,1\,\bar{1})$, $r_{,,}$ $(1\,2\,\bar{1})$. s is common to the zones ff', $r'r''$, therefore s is $(1\,3\,0)$. n is common to the zones $f''d''$, $r_{,}r_{,,}$, therefore n is $(3\,3\,\bar{2})$. The edges and solid angles formed by r, r', r'' are not truncated by any faces corresponding to p, n, therefore p, n belong to hemihedral forms with inclined faces. Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{1\,1\,0\}$, $\kappa\{1\,1\,\bar{1}\}$, $\{3\,1\,0\}$, $\{2\,1\,1\}$, $\kappa\{3\,3\,\bar{2}\}$.

CHAPTER III.

PYRAMIDAL SYSTEM.

80. In the pyramidal system the crystallographic axes make right angles with each other, and two of the parameters, a, b are equal.

81. The holohedral form $\{h\,k\,l\}$ is bounded by all the faces which have for their symbols the different arrangements of $\pm h$, $\pm k$, $\pm l$, in which l holds the last place. When h, k, l are all different they afford the sixteen arrangements contained in the annexed table. When one of the indices is zero, or when h, k are equal, the number will be eight. When l is zero and $h = k$, or when one of the indices h, k is zero, the number will be four. When h and k are zero it will be two.

$h\,k\,l$	$k\,h\,l$	$\bar{h}\,\bar{k}\,\bar{l}$	$\bar{k}\,\bar{h}\,\bar{l}$
$\bar{h}\,\bar{k}\,l$	$\bar{k}\,\bar{h}\,l$	$h\,k\,\bar{l}$	$k\,h\,\bar{l}$
$\bar{k}\,h\,\bar{l}$	$\bar{h}\,k\,\bar{l}$	$\bar{k}\,h\,l$	$\bar{h}\,k\,l$
$k\,\bar{h}\,\bar{l}$	$h\,\bar{k}\,\bar{l}$	$k\,\bar{h}\,l$	$h\,\bar{k}\,l$

If we suppose h to be greater than k, fig. 38 will represent the arrangement of the poles of the form $\{h\,k\,l\}$ on the surface of the sphere of projection.

82. The form bounded either by all the faces of $\{h\,k\,l\}$ which have an odd number of positive indices, or by all the faces of $\{h\,k\,l\}$ which have an odd number of negative indices, is said to be hemihedral with inclined faces, and will be denoted by the symbol $\kappa\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one

of its faces. The hemihedral form bounded by faces which have an odd number of positive indices is said to be direct. The form bounded by faces having an odd number of negative indices is said to be inverse. The symbols of the direct form are contained in the first and second columns, those of the inverse form in the third and fourth columns of the table in (81).

If the surface of the sphere of projection be divided into eight triangles by zone-circles through the poles of $\{0\,0\,1\}$ and $\{1\,0\,0\}$, the poles of the direct form will be found in four alternate triangles one of which contains the pole of $(1\,1\,1)$. The pole of the inverse form will be contained in the remaining four alternate triangles.

83. The pyramidal system admits of a second hemihedral form with inclined faces, which is bounded by all the faces of $\{h\,k\,l\}$, in which the order of h, k changes with the sign of l, and which will be denoted by the symbol $\kappa'\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one of its faces. The form is said to be direct or inverse, according as the first index is greater or less than the second, when the three indices have the same sign. The symbols of the faces of the direct form are contained in the first and fourth columns, those of the inverse form in the second and third columns of the table in (81).

If the surface of the sphere of projection be divided into eight triangles by great circles through the poles of $\{0\,0\,1\}$ and $\{1\,1\,0\}$, the poles of the direct form will be found in four alternate triangles, one of which contains the pole of $(1\,0\,1)$. The poles of the inverse form will be found in the remaining four alternate triangles.

84. The form bounded by all the faces of $\{h\,k\,l\}$, in which the order of h, k is the same or different according as h, k have the same or different signs, is said to be hemihedral with parallel faces, and will be denoted by the symbol $\pi\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one of its faces. The form is called direct or inverse according as the first index is greater or less than the second, when the two indices have

the same sign. The symbols of the faces of the direct form are contained in the first and third columns, those of the inverse form in the second and fourth columns of the table in (81).

If the surface of the sphere of projection be divided into eight lunes by zone-circles through the poles of (0 0 1) and the poles of {1 0 0}, {1 1 0}, the poles of the direct hemihedral form will be found in four alternate lunes, one of which is (1 0 0), (1 1 0); and those of the inverse form in the remaining four alternate lunes.

85. To find the position of any pole.

Let the axes of the crystal meet the surface of the sphere of projection in X, Y, Z (fig. 39). Let a, a, c be the parameters of the crystal; P the pole of $(h\,k\,l)$. Since the axes of the crystal are rectangular, YZ, ZX, XY are quadrants, therefore $\cos YZ = 0$, $\cos ZX = 0$, $\cos XY = 0$, therefore X, Y, Z are the poles of (1 0 0), (0 1 0), (0 0 1), and the angles at X, Y, Z right angles.

$$\cos PX = \sin PY \cos PYX = \sin PZ \cos PZX,$$
$$\cos PY = \sin PZ \cos PZY = \sin PX \cos PXY,$$
$$\cos PZ = \sin PX \cos PXZ = \sin PY \cos PYZ.$$

$$\cot PX = \tan PZY \cos PXY = \tan PYZ \cos PXZ,$$
$$\cot PY = \tan PXZ \cos PYZ = \tan PZX \cos PYX,$$
$$\cot PZ = \tan PYX \cos PZX = \tan PXY \cos PZY.$$

But

$$\frac{a}{h} \cos PX = \frac{a}{k} \cos PY = \frac{c}{l} \cos PZ.$$

Whence

$$\tan PXY = \frac{l}{k}\frac{a}{c}, \quad \tan PYX = \frac{l}{h}\frac{a}{c}, \quad \tan PZX = \frac{k}{h}.$$

$$\cot PX = \frac{h}{k} \cos PXY = \frac{h}{l} \frac{c}{a} \cos PXZ,$$

$$\cot PY = \frac{k}{h} \cos PYX = \frac{k}{l} \frac{c}{a} \cos PYZ,$$

$$\cot PZ = \frac{l}{h} \frac{a}{c} \cos PZX = \frac{l}{k} \frac{a}{c} \cos PZY,$$

$$(\tan PZ)^2 = \frac{c^2}{a^2} \frac{h^2 + k^2}{l^2}.$$

86. The poles of $\{1\,1\,0\}$ bisect the arcs joining any two adjacent poles of $\{1\,0\,0\}$. For if N be any pole of $\{1\,1\,0\}$; X, Y the adjacent poles of $\{1\,0\,0\}$, it will be found that $\cot NX = 1$, $\cot NY = 1$, therefore NX, NY are each 45^0, therefore N bisects XY.

87. It appears from the form of the expressions in (85), that the distances of the poles of $\{h\,k\,l\}$ from the nearest of the two poles $(0\,0\,1)$, $(0\,0\,\bar{1})$ are all equal; and that the angles subtended at $(0\,0\,1)$ or $(0\,0\,\bar{1})$ by the arcs joining any pole of $\{h\,k\,l\}$ and the nearest pole of $\{1\,0\,0\}$ are all equal. Hence, it may be easily shewn, that the poles of the form $\{h\,k\,l\}$ are symmetrically arranged with respect to each of the five zone-circles that can be drawn through every two of the poles of the three forms $\{0\,0\,1\}$, $\{1\,0\,0\}$, $\{1\,1\,0\}$.

88. If the surface of the sphere of projection be divided into sixteen triangles, by the five zone-circles drawn through the poles of every two of the forms $\{0\,0\,1\}$, $\{1\,0\,0\}$, $\{1\,1\,0\}$, the poles of $\{h\,k\,l\}$ will be symmetrically arranged with respect to any side of any one of the triangles. Hence the arrangement of the poles of $\{h\,k\,l\}$ will be symmetrical, in any two adjacent triangles, and similar in any two alternate triangles.

89. The poles of $\kappa\{h\,k\,l\}$ are symmetrically arranged with respect to the two zone-circles, drawn through the poles

of $\{0\,0\,1\}$ and the poles of $\{1\,1\,0\}$. The poles of $\kappa'\{h\,k\,l\}$ are symmetrically arranged with respect to the two zone-circles through the poles of $\{0\,0\,1\}$ and the poles of $\{1\,0\,0\}$. The poles of $\pi\{h\,k\,l\}$ are symmetrically arranged with respect to the zone-circle passing through the poles of $\{1\,0\,0\}$. The arrangement of the poles of $\pi\{h\,k\,l\}$ will be similar in any two triangles on the same side of the zone-circle through the poles of $\{1\,0\,0\}$, and symmetrical in any two triangles on opposite sides of it.

90. The direct and inverse forms differ only in position. For, if the sphere of projection be made to revolve through two right angles round any two opposite poles of $\{1\,0\,0\}$, the poles of the direct form will change places with the poles of the inverse form in the hemihedral form with parallel faces and in the first hemihedral form with inclined faces. And, in the second hemihedral form with inclined faces, if the sphere of projection be made to revolve through two right angles round any two opposite poles of $\{1\,1\,0\}$, the poles of the direct form will change places with the poles of the inverse form.

91. In the form $\{h\,k\,0\}$, if the distance between two poles be K, F or M, according as their symbols differ only in the sign of k, the order of the indices h, k, or in the order of the indices h, k, and sign of one of them, then (85)

$$\tan \tfrac{1}{2}K = \frac{k}{h}, \quad F = 90^0 - K, \quad M = 90^0.$$

92. In the form $\{h\,0\,l\}$, if L be the distance between two poles differing only in the sign of l, F the distance between two poles differing only in the arrangement of h, 0,

$$\tan \tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}, \quad \cos F = (\sin \tfrac{1}{2}L)^2.$$

93. In the form $\{h\,h\,l\}$, if K, L be the distances between two poles differing only in the signs of h, l respectively,

$$\tan \tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}\cos 45^0, \quad \cos K = (\sin \tfrac{1}{2}L)^2.$$

94. Let H, K, L be the distances between any two poles of the form $\{h\,k\,l\}$, differing only in the signs of h, k, l respectively. Let F be the distance between any two poles having the indices h, k in different order, and the signs of the first, second and third indices in one, the same as the signs of the first, second and third indices in the other. Let G be the distance between any two poles having the indices h, k in different order, the signs of the first and second indices in one, different from the signs of the first and second indices in the other, and the sign of the third index the same in both; and let M be the distance between any two poles differing only in the order of the indices h, k, and in the sign of one of them. Then $90^0 - \frac{1}{2}H$, $90^0 - \frac{1}{2}K$, $\frac{1}{2}L$ will be the distances of any pole of $\{h\,k\,l\}$ from the nearest two poles of $\{1\,0\,0\}$ and the nearest pole of $\{0\,0\,1\}$. F, G subtend at $(0\,0\,1)$ angles of $90^0 - 2\phi$, $90^0 + 2\phi$ respectively, where ϕ is the angle subtended at $(0\,0\,1)$ by the distance of any pole of $\{h\,k\,l\}$ from the nearest pole of $\{1\,0\,0\}$. M subtends an angle of 90^0 at $(0\,0\,1)$. Hence (85)

$$\text{if} \quad \tan\phi = \frac{k}{h}, \qquad \tan\tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}\cos\phi,$$

$$\sin\tfrac{1}{2}K = \cos\tfrac{1}{2}L\sin\phi, \quad \sin\tfrac{1}{2}H = \cos\tfrac{1}{2}L\cos\phi,$$

$$\sin\tfrac{1}{2}G = \cos\tfrac{1}{2}L\sin(45+\phi),$$

$$\sin\tfrac{1}{2}F = \cos\tfrac{1}{2}L\cos(45+\phi).$$

$$\cos M = (\sin\tfrac{1}{2}L)^2.$$

95. If the distance between two poles of either of the forms $\{h\,k\,0\}$, $\{h\,0\,l\}$, $\{h\,h\,l\}$ be given, the distance between the two poles, or its supplement, will be one of the arcs F, K, L, whence, from the expressions in (91)—(93), the ratio of the indices may be found.

96. If the distance between any pole of the form $\{h\,k\,l\}$ and each of two other poles of the same form be given, the

three poles not being in a great circle, the given distances, or their supplements, will be two of the arcs H, K, L, F, G, M, which being known, ϕ and L, and thence, from the expressions in (94), the ratios of the indices may be found.

97. Let P be the pole of $(h\,k\,l)$, Q the pole of $(p\,q\,r)$. Then (85)

$$\tan PX = \frac{h}{k}\cos PXY = \frac{h}{l}\frac{c}{a}\cos PXZ,$$

$$\tan QX = \frac{p}{q}\cos QXY = \frac{p}{r}\frac{c}{a}\cos QXZ.$$

Let Q be in the zone-circle PX. Then $QXY = PXY$, $QXZ = PXZ$, therefore

$$\frac{h}{p}\frac{\tan PX}{\tan QX} = \frac{k}{q} = \frac{l}{r}.$$

Similarly, when Q is in the zone-circle PY,

$$\frac{k}{q}\frac{\tan PY}{\tan QY} = \frac{l}{r} = \frac{h}{p}.$$

And when Q is in the zone-circle PZ,

$$\frac{l}{r}\frac{\tan PZ}{\tan QZ} = \frac{h}{p} = \frac{k}{q}.$$

98. Let P be the pole of $(h\,k\,l)$, Q the pole of $(p\,q\,r)$, Z the pole of $(0\,0\,1)$. Then (85)

$$(\tan PZ)^2 = \frac{c^2}{a^2}\frac{h^2+k^2}{l^2}, \quad (\tan QZ)^2 = \frac{c^2}{a^2}\frac{p^2+q^2}{r^2},$$

$$\therefore \quad \frac{l^2(\tan PZ)^2}{h^2+k^2} = \frac{r^2(\tan QZ)^2}{p^2+q^2}.$$

99. To find the distance between any two poles.

Let the axes of the crystal meet the surface of the sphere of projection in X, Y, Z, therefore (85) X is the

pole of $(1\,0\,0)$, Y the pole of $(0\,1\,0)$, Z the pole of $(0\,0\,1)$. Let P, Q (fig. 39) be the poles of $(h\,k\,l)$, $(p\,q\,r)$; and let PQ meet the zone-circle XY in M. Then, the symbol of M, and the tangents of MZX, PZX, QZX, PZM, QZM may be found in terms of h, k, l, p, q, r. PZ may be found in terms of a, c, h, k, l. $ZM=90^0$, $\therefore$ $\cos PM = \cos PZM \sin PZ$,

$$\tan QM : \tan PM = \tan QZM : \tan PZM.$$

Whence PQ, which is either the sum or difference of PM, QM, is known.

100. Having given the distance between any two poles, not both in the zone-circle $[1\,0\,0,\ 0\,1\,0]$, to find the ratio of the parameters a, c.

Let $(h\,k\,l)$, $(p\,q\,r)$ be the symbols of P, Q (fig. 39), and let $\tan PZM$, $\tan QZM$ be expressed in terms of h, k, l, p, q, r. Then

$$\tan QM : \tan PM = \tan QZM : \tan PZM, \text{ therefore}$$

$$\frac{\sin (QM + PM)}{\sin (QM - PM)} = \frac{\tan QZM + \tan PZM}{\tan QZM - \tan PZM}.$$

The given distance PQ is either $QM + PM$ or $QM - PM$, $\therefore$ PM, QM are both known. Cos $PM = \cos PZM \sin PZ$. PZ being known, the ratio of c to a is given by the equation

$$(\tan PZ)^2 = \frac{c^2}{a^2}\,\frac{h^2 + k^2}{l^2}.$$

101. To find the indices of any face when referred to the axes of the zones $[\bar{1}\,1\,0,\ 0\,0\,1]$, $[1\,1\,0,\ 0\,0\,1]$, $[1\,0\,0,\ 0\,0\,1]$ as crystallographic axes.

The symbols of the three zones are $[1\,1\,0]$, $[1\,\bar{1}\,0]$, $[0\,0\,1]$ respectively,

$\therefore$ (28) $e=1$, $f=1$, $g=0$, $h=1$, $k=-1$, $l=0$, $p=0$, $q=0$, $r=1$.

Hence, if u, v, w be the indices of any face when referred to the original axes, u', v', w' its indices when referred to the new axes,

$$u' = u + v, \ v' = u - v, \ w' = w.$$

102. To determine the figure and angles of the form $\{h\,k\,l\}$, when h, k, l take particular values.

The angle between normals to any two faces is obtained from the expressions in (91)...(94), and will be denoted by the letter which, in the accompanying figure, is placed upon the edge formed by the intersection of the given faces. The arrangement of the poles of the different forms is shewn in (fig. 38). The number of faces is given in (81).

103. The form $\{0\,0\,1\}$ has two faces $(0\,0\,1)$, $(0\,0\,\bar{1})$, which are parallel to each other.

104. The form $\{1\,0\,0\}$ (fig. 40) has four faces. Normals to any two adjacent faces of $\{1\,0\,0\}$, and to either face of $\{0\,0\,1\}$, make right angles with each other, therefore $F = 90^0$.

105. The form $\{1\,1\,0\}$ (fig. 41) has four faces. $\tan\frac{1}{2}K=1$, therefore $K = 90^0$.

A normal to any face of $\{1\,1\,0\}$ makes an angle of 45^0 with a normal to an adjacent face of $\{1\,0\,0\}$, and an angle of 90^0 with a normal to either face of $\{0\,0\,1\}$. For (85) the cotangents of the angles are 1 and 0 respectively.

106. The form $\{h\,k\,0\}$ (fig. 42) has eight faces.

$$\tan\tfrac{1}{2}K = \frac{k}{h}, \quad F = 90^0 - K.$$

In $\{2\,1\,0\}$, $\tan K = \dfrac{4}{3}$, $\therefore K = 53^0.7',8$, $\quad F = 36^0.52',2$.

In $\{3\,1\,0\}$, $\tan K = \dfrac{3}{4}$, $\therefore K = 36^0.52',2$, $\quad F = 53^0.7',8$.

In $\{3\ 2\ 0\}$, $\tan K = \frac{12}{5}$, $\therefore K = 67^0.22',8$, $F = 22^0.37',2$.

In $\{4\ 1\ 0\}$, $\tan K = \frac{24}{7}$, $\therefore K = 73^0.44',4$, $F = 16^0.15',6$.

In $\{5\ 1\ 0\}$, $\tan K = \frac{5}{12}$, $\therefore K = 22^0.37',2$, $F = 67^0.22',8$.

In $\{5\ 3\ 0\}$, $\tan K = \frac{15}{8}$, $\therefore K = 61^0.55',7$, $F = 28^0.4',3$.

In $\{7\ 1\ 0\}$, $\tan K = \frac{7}{24}$, $\therefore K = 16^0.15',6$, $F = 73^0.44',4$.

A normal to any face of $\{h\,k\,0\}$ makes with normals to either face of $\{0\,0\,1\}$, and the nearest faces of $\{1\,0\,0\}$, $\{1\,1\,0\}$, angles of 90^0, $\frac{1}{2}K$, $\frac{1}{2}F$ respectively.

107. The hemihedral form with parallel faces $\pi\{h\,k\,0\}$ is bounded by the alternate faces of $\{h\,k\,0\}$, the normals to which make right angles with each other.

108. The form $\{h\,0\,l\}$ (fig. 43) has eight faces.

$$\tan \tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}, \quad \cos F = (\sin \tfrac{1}{2}L)^2.$$

109. In the hemihedral form with inclined faces $\kappa'\{h\,0\,l\}$ (fig. 44),

$$U = 180^0 - L, \quad V = 180^0 - F.$$

110. The form $\{h\,h\,l\}$ (fig. 45) has eight faces.

$$\tan \tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}\cos 45^0, \quad \cos K = (\sin \tfrac{1}{2}L)^2.$$

111. In the hemihedral form with inclined faces $\kappa\{h\,h\,l\}$ (fig. 46),

$$W = 180^0 - L, \quad T = 180^0 - K.$$

112. Let P, Q be two adjacent poles of either of the forms $\{h\,h\,l\}$, $\{p\,0\,r\}$, equidistant from Z, the pole of $(0\,0\,1)$;

and let the arc of the zone-circle joining P, Q, contain S, a pole of the other form. Then SZ bisects the right angle PZQ, and the angle PSZ is a right angle,

$$\therefore \tan SZ = \cos 45^0 \tan PZ.$$

113. The form $\{h\, k\, l\}$ (fig. 47) has sixteen faces.

$$\text{If} \quad \tan \phi = \frac{k}{h}, \quad \tan \tfrac{1}{2}L = \frac{l}{h}\frac{a}{c} \cos \phi,$$

$$\sin \tfrac{1}{2}K = \cos \tfrac{1}{2}L \sin \phi, \quad \sin \tfrac{1}{2}F = \cos \tfrac{1}{2}L \cos (45 + \phi).$$

114. In the first hemihedral form with inclined faces $\kappa\{h\, k\, l\}$ (fig. 48), $T = 180^0 - H$,

$$\sin \tfrac{1}{2}G = \cos \tfrac{1}{2}L \sin (45 + \phi), \quad \cos \tfrac{1}{2}T = \cos \tfrac{1}{2}L \cos \phi.$$

115. In the second hemihedral form with inclined faces $\kappa'\{h\, k\, l\}$ (fig. 49), $V = 180^0 - G$,

$$\sin \tfrac{1}{2}H = \cos \tfrac{1}{2}L \cos \phi, \quad \cos \tfrac{1}{2}V = \cos \tfrac{1}{2}L \sin (45 + \phi).$$

116. In the hemihedral form with parallel faces $\pi\{h\, k\, l\}$ (fig. 50), the distance M subtends an angle of 90^0 at the pole of $(0\, 0\, 1)$, therefore

$$\cos M = (\sin \tfrac{1}{2}L)^2.$$

117. The cleavages, in crystals belonging to the pyramidal system, are parallel to the faces of one or more of the forms $\{0\, 0\, 1\}$, $\{1\, 0\, 0\}$, $\{1\, 1\, 0\}$, $\{h\, 0\, l\}$, $\{h\, h\, l\}$.

Examples.

118. In a crystal of Idocrase (fig. 51), the faces p, m, m' make right angles with each other. d is in the zone $m\,m'$, and makes equal angles with m, m'; therefore if m be $(1\,0\,0)$, m' $(0\,1\,0)$, p $(0\,0\,1)$, d will be $(1\,1\,0)$. In like manner d' will be $(1\,1\,0)$; c is in the zone pd, therefore its symbol will be the form $(h\,h\,l)$. Let the symbol of c be $(1\,1\,1)$. Similarly c' will be $(1\,\bar{1}\,1)$. v' is common to the zones pm', cm, therefore (15) v' is $(0\,1\,1)$. Similarly v is $(1\,0\,1)$. s is common to dc', mc, therefore s is $(3\,1\,1)$. w is common to dv, mc, therefore w is $(2\,1\,1)$. b is common to wm', pc, therefore b is $(2\,2\,1)$. e is common to mb, dc', therefore e is $(4\,2\,1)$. r is common to em', pc, therefore r is $(4\,4\,1)$. n is common to em', mc, therefore n is $(4\,1\,1)$. a is common to dv, cd', therefore a is $(3\,1\,2)$. h is common to ps, mm', therefore h is $(3\,1\,0)$. f is common to pw, mm', therefore f is $(2\,1\,0)$. Hence the crystal is a combination of the forms $\{0\,0\,1\}$, $\{1\,0\,0\}$, $\{1\,1\,1\}$, $\{1\,1\,0\}$, $\{2\,1\,0\}$, $\{3\,1\,0\}$, $\{2\,1\,1\}$, $\{3\,1\,1\}$, $\{4\,1\,1\}$, $\{2\,2\,1\}$, $\{4\,4\,1\}$, $\{3\,2\,1\}$. It is cleavable parallel to the faces of $\{0\,0\,1\}$, $\{1\,0\,0\}$, $\{0\,1\,1\}$.

Let m, p, c, &c. (fig. 52), be the poles of the faces m, p, c, &c. $pc = 37^0.7'$. The symbols of the poles are p $(0\,0\,1)$, m $(1\,0\,0)$, c $(1\,1\,1)$, d $(1\,1\,0)$, f $(2\,1\,0)$, h $(3\,1\,0)$, b $(2\,2\,1)$, r $(4\,4\,1)$, w $(2\,1\,1)$, s $(3\,1\,1)$, n $(4\,1\,1)$, a $(3\,1\,2)$, e $(4\,2\,1)$. Therefore (85), $\tan dm = 1$, $\tan fm = \frac{1}{2}$, $\tan hm = \frac{1}{3}$, therefore $dm = 45^0$, $fm = 26^0.34'$, $hm = 18^0.26'$. From (97) we have $\tan cp = \frac{1}{2}\tan bp = \frac{1}{4}\tan rp$; $\therefore\ bp = 56^0.33'$, $rp = 71^0.43'$. From (98) we have $\tan cp = \dfrac{2}{\sqrt{10}}\tan wp = \dfrac{1}{10}\tan ep =$

$$\frac{\sqrt{2}}{2}\tan vp = \frac{2}{\sqrt{5}}\tan ap = \frac{1}{\sqrt{5}}\tan sp = \frac{2}{\sqrt{17}}\tan cp.$$

$\therefore\ wp = 50^0.6'$, $ep = 67^0.19'$, $vp = 28^0.9'$, $ap = 40^0.14'$, $sp = 59^0.25'$. From (85) we have $\cos cm = \sin cp \cos 45^0$, $\cos bm$

$= \sin bp \cos 45^0$, $\cos rm = \sin rp \cos 45^0$, therefore $cm = 65^0.\,44\frac{1}{2}$, $bm = 53^0.\,51'$, $rm = 47^0.\,49'$, $\tan cm = 2 \tan wm = 3 \tan sm = 4 \tan nm$ (97). $\therefore$ $wm = 46^0.\,40'$, $sm = 35^0.14'\frac{1}{2}$, $nm = 27^0.55'$. $\tan bm = 2 \tan em$, therefore $em = 34^0.23'\frac{1}{2}$. $\tan wm' = 2 \tan bm'$, therefore $wm' = 69^0.\,56'$. $\tan rm' = \frac{1}{2} \tan em' = \frac{1}{4} \tan nm'$, $\therefore$ $em' = 65^0.37'\frac{1}{2}$, $nm' = 77^0.14'$. a, s are in the zone-circle hp, $hm = 18^0.26'$, $\therefore \cos am = \sin ap \cos 18^0.26'$, $\cos am' = \sin ap \cos 71^0.34'$, $\cos sm' = \sin sp \cos 71^0.34'$, $\therefore$ $am = 52^0.13'$, $am' = 78^0.13'$, $sm = 74^0.12'$. The distance between the poles of (0 0 1) and (1 1 1) is $37^0.\,7'$, therefore (1 0 0), a, a, c being the parameters of the crystal, if $a = 1$, $c = 0{,}53511$.

119. In a crystal of Anatase (fig. 53), the face c makes equal angles with each of the four faces p, p', p'', p''', and of these any two adjacent faces make the same angle with each other. We may therefore assume c to be (0 0 1), p (1 1 1), p' ($\bar{1}$ 1 1). v is in the zone cp, and s in the zone vp'. Let c, p, p', v, s be the poles of the faces c, p, p', v, s. Then, by measuring, it is found, that $pc = 68^0.18'$, $vc = 19^0.45'$, $sc = 25^0.30'$. Let the symbol of v be $(p\,q\,r)$, v is in the zone-circle pc,

$$\therefore \ (97)\ \frac{1}{r}\frac{\tan pc}{\tan vc} = \frac{1}{p} = \frac{1}{q}, \ \therefore p = 1,\ q = 1,\ r = 7\ ;\ \therefore v \text{ is } (1\ 1\ 7).$$

$$\text{Let } s \text{ be } (p\,q\,r),\ \therefore\ (98)\ \frac{r^2(\tan sc)^2}{p^2 + q^2} = \frac{(\tan pc)^2}{2},$$

$\therefore$ $26r^2 = 361\,(p^2 + q^2)$. s is in the zone-circle vp', the symbol of which is $[3\ 4\ \bar{1}]$, therefore (21) $3p + 4q - r = 0$. Whence $p = 5$, $q = 1$, $r = 19$. Hence the crystal is a combination of the forms $\{0\ 0\ 1\}$, $\{1\ 1\ 1\}$, $\{1\ 1\ 7\}$, $\{5\ \ 1\ \ 19\}$. The crystal is cleavable parallel to the faces of $\{1\ 1\ 1\}$. The parameters of the crystal being a, a, c, if $a = 1$, $c = 1{,}777$.

120. In a crystal of Copper Pyrites, (fig. 54), c, c', c'', c''' are the faces of a square pyramid. Let their symbols be c (1 1 1), c' ($\bar{1}$ 1 1), c'' ($\bar{1}$ $\bar{1}$ 1), c''' (1 $\bar{1}$ 1). p is in the zone cc''', and makes equal angles with c, c''', therefore (87), p is also in

the zone $[0\,0\,1,\ 1\,0\,0]$; therefore p is $(1\,0\,1)$. Similarly p' is $(0\,1\,\bar{1})$. $b_{\prime}$ is common to the zones $c'c'''$, pp'; therefore $b_{\prime}$ is $(\bar{1}\,1\,\bar{2})$. There is no face of $\{1\,0\,1\}$, between c and c', in the zone $c\,c'$, therefore p belongs to the hemihedral form with inclined faces $\kappa'\{1\,0\,1\}$. Hence the crystal is a combination of the forms $\{1\,1\,1\}$, $\{1\,1\,2\}$, $\kappa'\{1\,0\,1\}$. It is cleavable parallel to the faces of the forms $\{1\,1\,1\}$, $\{0\,0\,1\}$.

If we change the axes by the rule in (101), the symbols of the faces will become c $(0\,2\,1)$, $b_{\prime}$ $(\bar{1}\,0\,\bar{1})$, p $(1\,1\,1)$. By changing the axes, the hemihedral form, of which p is a face, becomes $\kappa\{1\,1\,1\}$.

121. In a crystal of Scheelate of Lime, (fig. 55), p is $(\bar{1}\,1\,1)$, n is $(0\,2\,1)$. p, g, n, a are in one zone. If p, g, n, a be the poles of p, g, n, a, it is found by measuring that $pg = 22^0.31'$, $pn = 39^0.40'$, $pa = 68^0.6'$.

Let X, Y, Z be the poles of $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$. Let pn meet the zone-circle XY in m. Then m will be $(1\,1\,0)$, and $pm = 90^0$. If $(u\,v\,w)$ be the symbol of any pole S in the zone-circle pm, substituting p, n, m and their indices, for P, Q, R and their indices, in (27), we have

$$\frac{\tan pS}{\tan pn} = \frac{v - w}{w}.$$

Hence, if $(u\,v\,w)$ be the symbol of g, we have $\frac{v}{w} = \frac{3}{2}$. The symbol of the zone-circle pm is $[1\,\bar{1}\,2]$, $\therefore$ (21) $u - v + 2w = 0$. Whence $u = -1$, $v = 3$, $w = 2$; therefore g is $(\bar{1}\,3\,2)$. In like manner a is $(2\,4\,1)$. Of the faces g, a, those only occur which have their poles in alternate lunes of the sphere of projection, therefore g, a belong to the hemihedral forms with parallel faces $\pi\{3\,1\,2\}$, $\pi\{2\,4\,1\}$.

CHAPTER IV.

RHOMBOHEDRAL SYSTEM.

122. In the rhombohedral system the axes make equal angles with each other, and the parameters are equal.

123. The holohedral form $\{h\,k\,l\}$ is bounded by all the faces which have for their symbols the different arrangements of $+h$, $+k$, $+l$, together with those of $-h$, $-k$, $-l$. When h, k, l are all different, the number of arrangements will be twelve, as shewn in the annexed table, except when the indices are 0, -1, 1. In this case, and also when two of the indices are equal, the number will be six, and when all three are equal, the number will be two.

$h\,k\,l$	$l\,k\,h$	$\bar{h}\,\bar{k}\,\bar{l}$	$\bar{l}\,\bar{k}\,\bar{h}$
$k\,l\,h$	$k\,h\,l$	$\bar{k}\,\bar{l}\,\bar{h}$	$\bar{k}\,\bar{h}\,\bar{l}$
$l\,h\,k$	$h\,l\,k$	$\bar{l}\,\bar{h}\,\bar{k}$	$\bar{h}\,\bar{l}\,\bar{k}$

If h be algebraically the greatest, and l the least of three unequal indices h, k, l, fig. 56 will represent the arrangement of the poles of the form $\{h\,k\,l\}$ on the surface of the sphere of projection.

124. The form bounded either by all the faces which have for their symbols the different arrangements of $+h$, $+k$, $+l$, or by all the faces which have for their symbols the different arrangements of $-h$, $-k$, $-l$, is said to be hemihedral with inclined symmetric faces, and will be denoted by the symbol $\kappa\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one of its faces. A form is said to be direct or inverse according as the alge-

braic sum of its indices is positive or negative. When the sum of the indices is zero, the form will be called direct or inverse according as the largest index is positive or negative. The symbols of the faces of the direct form are contained in the first and second columns, those of the inverse form in the third and fourth columns of the table in (123).

If the surface of the sphere of projection be divided into two hemispheres by the zone-circle through the poles of $\{0\,1\,\bar{1}\}$, the poles of the direct form, when the sum of its indices is finite, will be found in the hemisphere which contains the pole of $(1\,1\,1)$, and the poles of the inverse form in the other hemisphere. When the sum of the indices is zero, if the surface of the sphere of projection be divided into six lunes by great circles through the poles of $\{1\,1\,1\}$ and those of $\{0\,1\,\bar{1}\}$, the poles of the direct form will be found in three alternate lunes, one of which contains the pole of $(1\,0\,0)$, and the poles of the inverse form will be found in the remaining three alternate lunes.

125. The form bounded either by all the faces of $\{h\,k\,l\}$, the indices of which stand in the order $h\,k\,l\,h\,k$, or by all the faces of $\{h\,k\,l\}$, the indices of which stand in the order $l\,k\,h\,l\,k$, is said to be hemihedral with parallel faces, and will be denoted by the symbol $\pi\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one of its faces. The symbols of the faces are contained either in the first and third, or in the second and fourth columns of the table in (123).

If the surface of the sphere of projection be divided into twelve lunes by zone-circles through the poles of $\{1\,1\,1\}$, and those of each of the forms $\{2\,\bar{1}\,\bar{1}\}$, $\{0\,\bar{1}\,1\}$, the poles of $\pi\{h\,k\,l\}$ will be found in six alternate lunes, except when the algebraic sum of two of the indices is equal to twice the third. And if the sphere of projection be divided into twelve triangles by zone-circles through every two of the poles of $\{1\,1\,1\}$, $\{2\,\bar{1}\,\bar{1}\}$, the poles of $\pi\{h\,k\,l\}$ will be found in six alternate triangles, except when the sum of the indices is zero.

126. The form bounded either by all the faces of $\{h\,k\,l\}$ which have for their symbols the arrangements of $+h$, $+k$, $+l$, which stand in the order $h\,k\,l\,h\,k$, and those of $-h$, $-k$, $-l$, which stand in the order $l\,k\,h\,l\,k$, or by all the faces which have for their symbols the arrangements of $+h$, $+k$, $+l$ which stand in the order $l\,k\,h\,l\,k$, and those of $-h$, $-k$, $-l$ which stand in the order $h\,k\,l\,h\,k$, is said to be hemihedral with inclined asymmetric faces, and will be denoted by the symbol $a\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of any one of its faces. The symbols of the faces are contained either in the first and fourth, or in the second and third columns of the table in (123).

If the surface of the sphere of projection be divided into six lunes by zone-circles through the poles of $\{1\,1\,1\}$ and those of $\{2\,\bar{1}\,\bar{1}\}$, the poles of $a\{h\,k\,l\}$ will be found in three alternate lunes.

127. To determine the position of any pole.

Let the axes of the crystal meet the surface of the sphere of projection in X, Y, Z, (fig. 57), let O be the pole of $(1\,1\,1)$, P the pole of $(h\,k\,l)$. Since O is the pole of $(1\,1\,1)$, and $a = b = c$, $\cos OX = \cos OY = \cos OZ$, $\therefore\ OX = OY = OZ$. The axes make equal angles with each other, therefore $YZ = ZX = XY$. Hence YOZ, ZOX, XOY are each equal to 120^0;

$$\therefore \quad \cos POY - \cos POZ = \sqrt{3} \sin POX,$$

$$\cos POY + \cos POZ = -\cos POX.$$

$$\cos PX = \cos PO \cos XO + \sin PO \sin XO \cos POX,$$

$$\cos PY = \cos PO \cos YO + \sin PO \sin YO \cos POY,$$

$$\cos PZ = \cos PO \cos ZO + \sin PO \sin ZO \cos POZ.$$

Whence

$$\cos PY - \cos PZ = \sqrt{3} \sin PO \sin XO \sin POX,$$

$$\cos PY + \cos PZ = 2 \cos PO \cos XO - \sin PO \sin XO \cos POX,$$

$$\cos PX + \cos PY + \cos PZ = 3 \cos PO \cos XO,$$

$$3 \sin PO \sin XO \cos POX = 2 \cos PX - \cos PY - \cos PZ.$$

P is the pole of $(h\,k\,l)$, therefore

$$\frac{1}{h} \cos PX = \frac{1}{k} \cos PY = \frac{1}{l} \cos PZ.$$

Whence

$$\tan POX = \sqrt{3}\, \frac{k - l}{2h - k - l},$$

$$\tan PO \tan XO \cos POX = \frac{2h - k - l}{h + k + l}.$$

Similarly

$$\tan POY = \sqrt{3}\, \frac{l - h}{2k - l - h},$$

$$\tan PO \tan YO \cos POY = \frac{2k - l - h}{h + k + l},$$

and

$$\tan POZ = \sqrt{3}\, \frac{h - k}{2l - h - k},$$

$$\tan PO \tan ZO \cos POZ = \frac{2l - h - k}{h + k + l}.$$

$$(\tan PO)^2 (\tan XO)^2 = 4\, \frac{h^2 + k^2 + l^2 - kl - lh - hk}{(h + k + l)^2}.$$

128. Let A, B, C be the poles of $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$. Then (127) $\tan AOX = 0$, $\tan BOY = 0$, $\tan COZ = 0$, $\tan AO \tan XO = 2$, $\tan BO \tan YO = 2$, $\tan CO \tan ZO = 2$. Therefore A, B, C lie in the great circles OX, OY, OZ, $OA = OB = OC$, and the expressions in (127) become

$$\tan POA = \sqrt{3}\,\frac{k - l}{2h - k - l},$$

$$2 \tan PO \cot AO \cos POA = \frac{2h - k - l}{h + k + l},$$

$$\tan POB = \sqrt{3}\,\frac{l - h}{2k - l - h},$$

$$2 \tan PO \cot BO \cos POB = \frac{2k - l - h}{h + k + l},$$

$$\tan POC = \sqrt{3}\,\frac{h - k}{2l - h - k},$$

$$2 \tan PO \cot CO \cos POC = \frac{2l - h - k}{h + k + l},$$

$$\left(\frac{\tan PO}{\tan AO}\right)^2 = \frac{h^2 + k^2 + l^2 - kl - lh - hk}{(h + k + l)^2} = \frac{\overline{k-l}^2 + \overline{l-h}^2 + \overline{h-k}^2}{2(h+k+l)^2}$$

129. If M, N be any poles of the forms $\{2\,\bar{1}\,\bar{1}\}$, $\{0\,1\,\bar{1}\}$; A any pole of $\{1\,0\,0\}$, O a pole of $\{1\,1\,1\}$, the expressions in (128) shew that MO, NO are quadrants; that MOA is a multiple of 60°, and that NOA is an odd multiple of 30°. Hence the poles of $\{2\,\bar{1}\,\bar{1}\}$ are six equidistant points in which a zone-circle having $(1\,1\,1)$ for one of its poles, is intersected by the zone-circles through the poles of $\{1\,1\,1\}$ and those of $\{1\,0\,0\}$; and the poles of $\{0\,1\,\bar{1}\}$ bisect the arcs joining every two adjacent poles of $\{2\,\bar{1}\,\bar{1}\}$.

130. From the form of the expression for $\tan PO$, it appears that the distances of the poles of $\{h\,k\,l\}$, which have the indices $+h$, $+k$, $+l$, from $(1\,1\,1)$, and of those which have the indices $-h$, $-k$, $-l$, from $(\bar{1}\,\bar{1}\,\bar{1})$, are all equal. By interchanging the indices h, k, l, and changing their signs in the expressions for $\tan POA$, $\tan POB$, $\tan POC$, it appears that the angles subtended at $(1\,1\,1)$, by the arcs joining any pole of $\{h\,k\,l\}$ and the nearest pole of $\{1\,0\,0\}$, are all equal. Hence, the poles of $\{h\,k\,l\}$ are symmetrically arranged with respect to each of the three zone-circles passing through the poles of $\{1\,1\,1\}$ and those of $\{2\,\bar{1}\,\bar{1}\}$.

131. If the surface of the sphere of projection be divided into twelve triangles, by zone-circles through every two of the poles of $\{1\,1\,1\}$ and $\{2\,\bar{1}\,\bar{1}\}$, the arrangement of the poles of $\{h\,k\,l\}$ will be symmetrical in any two adjacent triangles on the same side of the zone-circle $[2\,\bar{1}\,\bar{1},\,\bar{1}\,\bar{1}\,2]$, or in any two alternate triangles on different sides of it; and similar in any two adjacent triangles on different sides of the zone-circle $[2\,\bar{1}\,\bar{1},\,\bar{1}\,\bar{1}\,2]$, or in any two alternate triangles on the same side of it.

132. The arrangement of the poles of $\kappa\{h\,k\,l\}$ will be symmetrical in any two adjacent lunes, composed of two adjacent triangles on different sides of $[2\,\bar{1}\,\bar{1},\,\bar{1}\,\bar{1}\,2]$, and similar in any two alternate lunes. The arrangement of the poles of $\pi\{h\,k\,l\}$ in any two triangles is similar or symmetrical, according as the triangles are on the same side of the zone-circle $[2\,\bar{1}\,\bar{1},\,\bar{1}\,\bar{1}\,2]$ or on different sides of it. The poles of $a\{h\,k\,l\}$ are similarly arranged in each of the triangles in which they occur.

133. If P be the pole of any face, and if in PO produced OQ be taken equal to OP, a face may always exist of which Q is the pole.

We have from (127)

$$2\cos QX - \cos QY - \cos QZ = 3\sin QO \sin OX \cos QOX,$$

$$\cos QX + \cos QY + \cos QZ = 3\cos QO \cos OX,$$

$$\cos QZ - \cos QY = \sqrt{3}\sin QO \sin OX \sin QOX,$$

$$\sin QO = \sin PO,\quad \cos QOX = -\cos POX,\quad \sin QOX = -\sin POX.$$

Hence, if $(h\,k\,l)$ be the symbol of P,

$$\frac{2\cos QX - \cos QY - \cos QZ}{\cos QX + \cos QY + \cos QZ} = -\frac{2h - k - l}{h + k + l},$$

$$\frac{\cos QZ - \cos QY}{\cos QX + \cos QY + \cos QZ} = -\frac{l - k}{h + k + l}.$$

Whence

$$\frac{3\cos QX}{\cos QX + \cos QY + \cos QZ} = \frac{-h+2k+2l}{h+k+l},$$

$$\frac{3\cos QY}{\cos QX + \cos QY + \cos QZ} = \frac{2h-k+2l}{h+k+l},$$

$$\frac{3\cos QZ}{\cos QX + \cos QY + \cos QZ} = \frac{2h+2k-l}{h+k+l};$$

$$\therefore \ \frac{1}{p}\cos QX = \frac{1}{q}\cos QY = \frac{1}{r}\cos QZ,$$

where

$$p = -h+2k+2l, \ q = 2h-k+2l, \ r = 2h+2k-l.$$

p, q, r are whole numbers, therefore a face may exist of which Q is the pole.

134. When h, k, l, p, q, r are connected by the equations

$$p = -h+2k+2l, \ q = 2h-k+2l, \ r = 2h+2k-l,$$

and, therefore, the arc joining the poles $(h\,k\,l)$, $(p\,q\,r)$, is bisected by $(1\,1\,1)$, the forms $\{h\,k\,l\}$, $\{p\,q\,r\}$ are said to be transverse with respect to each other. In certain crystals belonging to the rhombohedral system, combinations of pairs of transverse forms occur frequently. Such a combination is termed dirhombohedral. The arrangements of its poles is shewn in fig. 58.

135. If the surface of the sphere of projection be divided into twenty-four triangles by zone-circles through every two of the poles of $\{1\,1\,1\}$, $\{0\,1\,\bar{1}\}$, $\{2\,\bar{1}\,\bar{1}\}$, the poles of the dirhombohedral combination $\{h\,k\,l\}$, $\{p\,q\,r\}$, or $\kappa\{h\,k\,l\}$, $\kappa\{p\,q\,r\}$, will be symmetrically arranged in any two adjacent triangles, and similarly arranged in any two alternate triangles. The poles of the dirhombohedral combination $\pi\{h\,k\,l\}$, $\pi\{p\,q\,r\}$ will be similarly or symmetrically arranged in any two triangles, according as the triangles are on the same side of the

zone-circle $[2\,\bar{1}\,\bar{1},\ \bar{1}\,\bar{1}\,2]$, or on different sides of it. The poles of the dirhombohedral combination $a\{h\,k\,l\}$, $a\{p\,q\,r\}$ are similarly arranged in all the triangles in which they are found.

136. In the hemihedral forms with parallel faces, and in the hemihedral forms with inclined asymmetric faces, the faces of different forms of the same kind occasionally occur in zones. If MQM' (fig. 59) be the zone-circle of one of these zones; MLM' the zone-circle through the poles of $\{2\,\bar{1}\,\bar{1}\}$, and QML an acute angle, the zone is called direct or inverse, according as the poles of its faces lie nearer to M or M'. The zones in a combination of hemihedral forms with parallel faces are direct on one side of the zone-circle through the poles of $\{2\,\bar{1}\,\bar{1}\}$, and inverse on the other side of it. In a combination of hemihedral forms with inclined asymmetric faces, the zones are either all direct or all inverse, and the combination is named accordingly.

137. The direct and inverse hemihedral forms with inclined symmetric faces, and the two hemihedral forms with parallel faces differ only in position. For if the sphere of projection be made to revolve through two right angles round any two opposite poles of $\{0\,1\,\bar{1}\}$, the poles of one half form change places with those of the other half form. The direct and inverse hemihedral forms with inclined asymmetric faces are essentially different.

138. Let P, A be any two adjacent poles of $\{h\,k\,k\}$, $\{1\,0\,0\}$, O the nearest pole of $\{1\,1\,1\}$. Then (128) $\tan POA = 0$, therefore P, A, O are in one zone-circle. Let $PO = T$, $AO = D$. Therefore, making $l = k$ in (128), we have

$$\tan T = \frac{h - k}{h + 2k} \tan D.$$

The signs of $\tan T$, $\tan D$, will be the same or different according as the directions in which PO, AO are measured from O are the same or different.

If V be the distance between any two of three poles of $\{h\,k\,k\}$, which have the indices h, k, k, V subtends an angle of 120^0 at (1 1 1); and the two sides that include the angle of 120^0 are each equal to T, therefore

$$\sin \tfrac{1}{2} V = \sin 60 \sin T.$$

The distance between any two adjacent poles one of which has the indices $+h$, $+k$, $+l$, and the other the indices $-h$, $-k$, $-l$, will be $180^0 - V$.

139. If P be any pole of $\{h\,k\,l\}$, where $h + k + l = 0$, A the nearest pole of $\{1\,0\,0\}$, $PO = 90^0$ (128). Therefore, if H be the distance between any two adjacent poles of $\{h\,k\,l\}$ on opposite sides of a pole $2\,\bar{1}\,\bar{1}$ having the indices h, k, l, $\tfrac{1}{2}H = POA$, therefore

$$\tan \tfrac{1}{2} H = \sqrt{3}\,\frac{k - l}{2h - k - l}.$$

140. Let the arc joining any two poles of $\{h\,k\,l\}$, having the indices $+h$, $+k$, $+l$, be H, K, L or V, according as h, k, l or neither of the indices holds the same place in the symbols of the two poles; T the distance of either of the poles from (1 1 1); D the distance of any pole of $\{1\,0\,0\}$ from the nearest pole of $\{1\,1\,1\}$; 2θ, 2ϕ, 2ψ the angles subtended at (1 1 1) by H, K, L. The angle subtended by V will be 120^0. Then (128)

$$(\tan T)^2 = \frac{h^2 + k^2 + l^2 - kl - lh - hk}{(h + k + l)^2}(\tan D)^2,$$

$$\tan \theta = \sqrt{3}\,\frac{k - l}{2k - k - l}, \qquad /h$$

$$\tan \phi = \sqrt{3}\,\frac{l - h}{2k - l - h},$$

$$\tan \psi = \sqrt{3}\,\frac{h - k}{2l - h - k}.$$

The triangles having the vertex (1 1 1), and bases H, K, L, V are isosceles, therefore

$$\sin \tfrac{1}{2}H = \sin\theta \sin T, \quad \sin \tfrac{1}{2}K = \sin\phi \sin T,$$
$$\sin \tfrac{1}{2}L = \sin\psi \sin T, \quad \sin \tfrac{1}{2}V = \sin 60^\circ \sin T.$$

141. If V, the distance between any two of three equidistant poles of $\{h\,k\,k\}$, be given, we have

$$\sin \tfrac{1}{2}V = \sin 60^\circ \sin T, \ \tan T = m \tan D,$$
$$(h-k) = \pm m\,(h+2k) \quad (138),$$

the upper or lower sign being taken according as T and D are measured from (1 1 1) in the same, or in different directions. Whence, m being known, the ratio of h to k may be found.

142. In the form $\{h\,k\,l\}$, where $h+k+l=0$, the distance between any two poles, not a multiple of 60°, being known, we can find the distance of one of them from the nearest pole of $\{2\,\bar{1}\,\bar{1}\}$. If this distance be θ, the ratios of h, k, l may be found from the equations

$$\tan\theta = \sqrt{3}\,\frac{k-l}{2h-k-l}, \quad h+k+l=0.$$

143. If the distances between any pole of the form $\{h\,k\,l\}$ and each of two other poles of the same form be given, the three poles not being in one zone-circle, the given distances, or their supplements, will be two of the arcs H, K, L, V. By eliminating T between the equations for determining H, K, L, V in (140), observing that $\phi-\theta=60^\circ$, $\psi+\theta=60^\circ$, we have

$$\frac{\tan\theta}{\tan 60^\circ} = \frac{\tan\frac{1}{4}(K-L)}{\tan\frac{1}{4}(K+L)}, \quad \frac{\sin\theta}{\sin 60^\circ} = \frac{\sin\frac{1}{2}H}{\sin\frac{1}{2}V},$$

$$\frac{\tan\phi}{\tan 60^\circ} = \frac{\tan\frac{1}{4}(L+H)}{\tan\frac{1}{4}(L-H)}, \quad \frac{\sin\phi}{\sin 60^\circ} = \frac{\sin\frac{1}{2}K}{\sin\frac{1}{2}V},$$

$$\frac{\tan\psi}{\tan 60^\circ} = \frac{\tan\frac{1}{4}(K-H)}{\tan\frac{1}{4}(K+H)}, \quad \frac{\sin\psi}{\sin 60^\circ} = \frac{\sin\frac{1}{2}L}{\sin\frac{1}{2}V}.$$

Two of the four distances H, K, L, V being known, T and one of the angles θ, ϕ, ψ may be found; and then the ratios of the indices may be found from the equations

$$\tan\theta = \sqrt{3}\,\frac{k-l}{2h-k-l}, \quad 2\tan T\cos\theta = \frac{2h-k-l}{h+k+l}\tan D,$$

$$\tan\phi = \sqrt{3}\,\frac{l-h}{2k-l-h}, \quad 2\tan T\cos\phi = \frac{2k-l-h}{h+k+l}\tan D,$$

$$\tan\psi = \sqrt{3}\,\frac{h-k}{2l-h-k}, \quad 2\tan T\cos\psi = \frac{2l-h-k}{h+k+l}\tan D.$$

144. To find the distance between any two poles.

Let P, Q (fig. 60) be the poles of $(h\,k\,l)$, $(p\,q\,r)$; O, A poles of $\{1\,1\,1\}$, $\{1\,0\,0\}$. Let PQ meet the zone-circle $[0\,1\,\bar{1},\ \bar{1}\,0\,1]$ in M. Then, the symbol of M, and the tangents of MOA, POA, QOA, POM, QOM may be found in terms of h, k, l, p, q, r; and PO may be found in terms of AO, h, k, l. MO is a quadrant, therefore

$$\cos PM = \cos POM \sin PO,$$

$$\frac{\tan QM}{\tan PM} = \frac{\tan QOM}{\tan POM}.$$

Whence PQ, which is either the sum or difference of PM, QM, is known.

145. Having given the distance between any two poles, not both in the zone-circle $[0\,1\,\bar{1},\ \bar{1}\,0\,1]$, to find D, the distance of any pole of $\{1\,0\,0\}$ from the nearest pole of $\{1\,1\,1\}$.

Let P, Q (fig. 60) be the given poles, $(h\,k\,l)$, $(p\,q\,r)$ their symbols, then, retaining the construction in (144), let $\tan POM$, $\tan QOM$ be expressed in terms of h, k, l, p, q, r.

$$\frac{\tan QM}{\tan PM} = \frac{\tan QOM}{\tan POM},$$

therefore

$$\frac{\sin(QM + PM)}{\sin(QM - PM)} = \frac{\tan QOM + \tan POM}{\tan QOM - \tan POM}.$$

One of the arcs $QM - PM$, $QM + PM$ is the given distance PQ, therefore PM, QM are both known.

$$\cos PM = \cos POM \sin PO.$$

PO being known, D is given by the equation

$$(\tan PO)^2 = \frac{h^2 + k^2 + l^2 - kl - lh - hk}{(h + k + l)^2} (\tan D)^2.$$

146. To determine the position of any pole, having given its distances from two of three equidistant poles of any form.

Let P (fig. 61) be the given pole; A, B, C three equidistant poles of any known form; O the pole of $(1\,1\,1)$. Let AB meet the zone-circles $[0\,1\,\bar{1}, \bar{1}\,0\,1]$, CO in M, E, and let PM meet CO in N. AB is bisected in E; and ME, MN are perpendicular to EO.

$$\sin AE = \sin 60^0 \sin AO, \quad \tan EO = \cos 60^0 \tan AO,$$

$$\cos PA = \cos AE \cos PE + \sin AE \sin PE \cos PEA,$$

$$\cos PB = \cos BE \cos PE + \sin BE \sin PE \cos PEB.$$

If we express $\cos PA - \cos PB$, $\cos PA + \cos PB$ in terms of products of trigonometrical ratios, observing that $BE = AE$, $-\cos PEB = \cos PEA = \sin PEN$, and that $\sin PN = \sin PE \sin PEN$, we obtain

$$\sin AE \sin PN = \sin \tfrac{1}{2}(PB + PA) \sin \tfrac{1}{2}(PB - PA),$$

$$\cos AE \cos PE = \cos \tfrac{1}{2}(PB + PA) \cos \tfrac{1}{2}(PB - PA).$$

Whence, PE, PN being known, NE and therefore NO may be found.

$$\cos PO = \cos PN \cos NO, \quad \cot PON = \sin NO \cot PN.$$

147. To find the indices of any face when referred to the axes of the zones containing the faces $(h\,k\,k)$, $(k\,h\,k)$, $(k\,k\,h)$, as crystallographic axes.

The indices of the three zones reduced to their simplest terms are

$$h+k,\ -k,\ -k\ ; \quad -k,\ h+k,\ -k\ ; \quad -k,\ -k,\ h+k,$$

therefore (28), $\mathrm{e} = h+k$, $\mathrm{f} = -k$, $\mathrm{g} = -k$, $\mathrm{h} = -k$, $\mathrm{k} = h+k$, $\mathrm{l} = -k$, $\mathrm{p} = -k$, $\mathrm{q} = -k$, $\mathrm{r} = h+k$. Therefore, if u, v, w be the indices of any face when referred to the original axes, u', v', w', its indices when referred to the new axes,

$$u' = (h+k)\,u - kv - kw,$$

$$v' = -ku + (h+k)\,v - kw,$$

$$w' = -ku - kv + (h+k)w.$$

148. To determine the figure and angles of the form $\{h\,k\,l\}$, when h, k, l take particular values.

The angle between normals to any two faces of the same form may be computed by the formulæ in (138), (139), (140), and will be denoted by the letter placed upon the corresponding edge of the accompanying figure. The arrangement of the poles, when the three indices are unequal, is shewn in fig. 56. The poles of $\{h\,k\,k\}$ lie in zone-circles through the poles of $\{1\,1\,1\}$ and those of $\{2\,\bar{1}\,\bar{1}\}$. The number of faces is given in (123).

149. The form $\{1\,1\,1\}$ has two parallel faces. A normal to the faces makes equal angles with the three axes.

150. The hemihedral forms $\kappa\{1\,1\,1\}$, $\kappa\{\bar{1}\,\bar{1}\,\bar{1}\}$ consist of the faces $(1\,1\,1)$, $(\bar{1}\,\bar{1}\,\bar{1})$, respectively.

151. The form $\{h\,k\,k\}$ has six faces, and is called a rhombohedron. The three poles which have the indices $+h$, $+k$, $+k$ are equidistant from each other, and are dia-

metrically opposite to the three poles having the indices $-h$, $-k$, $-k$. Hence, a rhombohedron is bounded by three pairs of parallel faces making equal angles with each other. Let T be the distance of any pole of $\{h\,k\,k\}$ from the nearest pole of $\{1\,1\,1\}$; V the distance between two adjacent poles equidistant from $(1\,1\,1)$; W the distance between two adjacent poles not equally distant from $(1\,1\,1)$, and D the distance of a pole of $\{1\,0\,0\}$ from the nearest pole of $\{1\,1\,1\}$. Then (138)

$$\tan T = \frac{h-k}{h+2k} \tan D,$$

$$\sin \tfrac{1}{2} V = \sin 60^0 \sin T, \quad W = 180^0 - V.$$

The position of the rhombohedron $\{h\,k\,k\}$ is said to be parallel or transverse according as $\tan T$, $\tan D$ have the same or different signs, that is, according as T, D are measured from $(1\,1\,1)$ in the same direction or in different directions.

If we take fig. 62 to represent $\{1\,0\,0\}$, then $\{0\,1\,1\}$, which is in transverse position, will resemble fig. 63. In $\{0\,1\,1\}$, $\tan T = -\frac{1}{2} \tan D$. The poles of $\{0\,1\,1\}$ bisect the arcs joining the adjacent poles of $\{1\,0\,0\}$.

In $\{2\,1\,1\}$ which is in parallel position, $\tan T = \frac{1}{4} \tan D$. The poles of $\{2\,1\,1\}$ bisect the arcs joining the adjacent poles of $\{0\,1\,1\}$.

In $\{3\,1\,1\}$, which is in parallel position, $\tan T = \frac{2}{5} \tan D$.

In $\{\bar{1}\,2\,2\}$, which is in transverse position, $\tan T = -\tan D$. Hence, position excepted, the form of $\{\bar{1}\,2\,2\}$ is the same as that of $\{1\,0\,0\}$.

In $\{\bar{1}\,1\,1\}$ (fig. 64), which is in transverse position, $\tan T = -2 \tan D$. The poles of $\{1\,0\,0\}$ bisect the arcs joining the adjacent poles of $\{\bar{1}\,1\,1\}$.

In $\{3\,\bar{1}\,\bar{1}\}$, which is in parallel position, $\tan T = 4 \tan D$. The poles of $\{\bar{1}\,1\,1\}$ bisect the arcs joining the adjacent poles of $\{3\,\bar{1}\,\bar{1}\}$.

152. The hemihedral form with inclined symmetric faces $\kappa\{h\,k\,k\}$, has the three faces having the indices $+h$, $+k$, $+k$, and which make equal angles with each other. The other half form, has the three faces with the indices $-h$, $-k$, $-k$.

153. Let P, Q, R be three poles of a rhombohedron, equidistant from O, the pole of $(1\,1\,1)$; and let the zone-circle through P, Q, contain S, a pole of another rhombohedron. S is in the zone-circle RO which bisects the angle POQ, and the arc PQ. $POQ = 120^0$, therefore $POS = 60^0$; $PSO = 90^0$. Whence $\tan PO = 2 \tan SO$.

154. The form $\{\bar{2}\,1\,1\}$ has six faces, the poles of which (129) are the six equidistant points in which a great circle, having $(1\,1\,1)$, $(\bar{1}\,\bar{1}\,\bar{1})$ for its poles, is intersected by the zone-circles through the poles of $\{1\,1\,1\}$, and those of $\{1\,0\,0\}$. Therefore the distance between any two adjacent poles of $\{\bar{2}\,1\,1\}$ is 60^0.

155. The hemihedral form with inclined symmetric faces $\kappa\{\bar{2}\,1\,1\}$ is bounded by the alternate faces of $\{\bar{2}\,1\,1\}$. The distance between any two of its poles will be 120^0.

156. The form $\{0\,\bar{1}\,1\}$ has six faces the poles of which (129) bisect the arcs joining every two adjacent poles of $\{\bar{2}\,1\,1\}$. The distance between any two adjacent poles of $\{0\,\bar{1}\,1\}$ is 60^0.

157. The form $\{h\,k\,l\}$, where $h + k + l = 0$, (fig. 65) has twelve faces, the poles of which lie in the zone-circle through the poles of $\{\bar{2}\,1\,1\}$.

If H be the distance between any two poles adjacent to a pole of $\{\bar{2}\,1\,1\}$, h the largest index,

$$\tan \tfrac{1}{2}H = \sqrt{3}\,\frac{k - l}{2h - k - l}, \quad G = 60^0 - H.$$

The distances of any pole from the adjacent poles of $\{\bar{2}\,1\,1\}$, $\{0\,\bar{1}\,1\}$ are $\frac{1}{2}H$, $\frac{1}{2}G$ respectively.

In $\{2\,1\,\bar{3}\}$, $\tan\frac{1}{2}H = \frac{1}{9}\sqrt{3}$, $\therefore H = 21^0.\,47',2$.

In $\{3\,1\,4\}$, $\tan\frac{1}{2}H = \frac{1}{6}\sqrt{3}$, $\therefore H = 32^0.\,12',3$.

In $\{4\,1\,\bar{5}\}$, $\tan\frac{1}{2}H = \frac{1}{5}\sqrt{3}$, $\therefore H = 38^0.\,12',8$.

In $\{3\,2\,\bar{5}\}$, $\tan\frac{1}{2}H = \frac{1}{15}\sqrt{3}$, $.\ H = 13^0.\,10',4$.

In $\{5\,1\,\bar{6}\}$, $\tan\frac{1}{2}H = \frac{2}{9}\sqrt{3}$, $\therefore H = 42^0.\,6',4$.

In $\{5\,2\,\bar{7}\}$, $\tan\frac{1}{2}H = \frac{1}{7}\sqrt{3}$, $\therefore H = 27^0.\,47',7$.

In $\{7\,1\,\bar{8}\}$, $\tan\frac{1}{2}H = \frac{1}{4}\sqrt{3}$, $\therefore H = 46^0.\,49',6$.

158. In the hemihedral form with inclined symmetric faces $\kappa\{h\,k\,l\}$, where $h + k + l = 0$, the distances between the poles of two adjacent faces, are alternately H and $120^0 - H$.

159. The hemihedral form with parallel faces $\pi\{h\,k\,l\}$, is bounded by the alternate faces of $\{h\,k\,l\}$. The distance between any two adjacent poles is 60^0.

160. The form $\{h\,k\,l\}$ (fig. 66) has twelve faces. Let h, l be algebraically the greatest and least of the three indices. Then (140), if T, D be the distances of any poles of $\{h\,k\,l\}$, $\{1\,0\,0\}$ respectively from the nearest poles of $\{1\,1\,1\}$,

$$(\tan T)^2 = \frac{h^2 + k^2 + l^2 - kl - lh - hk}{(h + k + l)^2}(\tan D)^2,$$

$$\tan\theta = \sqrt{3}\,\frac{k - l}{2h - k - h},$$

$$\tan\phi = \sqrt{3}\,\frac{l - h}{2k - l - h},$$

$$\tan\psi = \sqrt{3}\,\frac{h - k}{2l - h - k},$$

$\sin\frac{1}{2}H = \sin\theta\sin T$, $\sin\frac{1}{2}K = \sin\phi\sin T$, $\sin\frac{1}{2}L = \sin\psi\sin T$,

G, F will be respectively equal to the greatest and least of the two angles H, L, and $W = 180^0 - K$.

When the algebraic sum of two of the indices is equal to twice the third, $\theta = \psi$, therefore $G = F$.

161. The hemihedral form with inclined symmetric faces $\kappa\{h\,k\,l\}$, has the faces of one of the two pyramids which, joined base to base, constitute the holohedral form $\{h\,k\,l\}$.

162. The hemihedral form with parallel faces $\pi\{h\,k\,l\}$ is bounded by the alternate faces of $\{h\,k\,l\}$ which occur in three parallel pairs making equal angles with each other. If the distances between two adjacent poles equally and unequally distant from (1 1 1) be V, W,

$$V = \sin 60^0 \sin T, \quad W = 180^0 - V.$$

163. In the hemihedral form with inclined asymmetric faces $a\{h\,k\,l\}$, if the distance between adjacent poles equally distant from (1 1 1), be V; and if the distances between adjacent poles enequally distant from (1 1 1) be U, W,

$$\sin \tfrac{1}{2} V = \sin 60^0 \sin T, \quad U = 180 - H, \quad W = 180 - K.$$

164. The cleavages, in crystals belonging to the rhombohedral system, are parallel to the faces of forms which have two or all of their indices equal.

EXAMPLES.

165. In a crystal of Calc Spar (fig. 67), the faces p, p', p'' are parallel to three cleavage planes, the poles of which are distant $74^0.55'$ from each other. Let p be $(1\,0\,0)$, p' $(0\,1\,0)$, p'' $(0\,0\,1)$. Let p, p', &c. (fig. 70), be the poles of p, p', &c. g' is in the zone-circle pp'', and bisects the arc pp''; therefore g' is in the zone-circle $p'o$, o being the pole of $(1\,1\,1)$, therefore g' is $(1\,0\,1)$. c' is one of six faces in a zone, the axis of which makes equal angles with normals to p, p', p''. c' is also in the zone $p'g'$, therefore c' is $(1\,\bar{2}\,1)$. In like manner c is $(2\,\bar{1}\,\bar{1})$. Let f be the pole of $(h\,k\,k)$, then f' will be the pole of $(k\,h\,k)$, and the zone-circle ff' will intersect the zone-circle cc' in e', the pole of $(1\,\bar{1}\,0)$. p'' is in the zone-circle ff', therefore f' is the intersection of $e'p''$, $p'g'$, therefore f' is $(1\,\bar{1}\,1)$. cp', pp'', intersect in r, therefore r is $(2\,0\,\bar{1})$. $g'c_{,,}$, pp' intersect in t, therefore t is $(3\,1\,0)$, t' $(3\,0\,1)$. tt', $p''c''$ intersect in ϕ'', therefore ϕ'' is $(3\,3\,\bar{2})$. Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{0\,1\,1\}$, $\{1\,1\,\bar{1}\}$, $\{2\,\bar{1}\,\bar{1}\}$, $\{3\,3\,\bar{2}\}$, $\{3\,1\,0\}$, $\{2\,0\,\bar{1}\}$.

166. In a crystal of Calc Spar (fig. 68) the faces p, p', p'' are parallel to the three cleavage planes. Let p be $(1\,0\,0)$, p' $(0\,1\,0)$, p'' $(0\,0\,1)$; and let p, p', &c. (fig. 70) be the poles of p, p', &c. o is equidistant from p, p', p'', therefore (128), o is the pole of $(1\,1\,1)$. $oc_{,,} = 90^0$, and $c_{,,}$ is in the zone-circle op'', therefore $c_{,,}$ is $(1\,1\,\bar{2})$. c is $(2\,\bar{1}\,\bar{1})$. cp', pp'' intersect in r, therefore r is $(2\,0\,\bar{1})$. r' is $(2\,\bar{1}\,0)$, and r'' is $(0\,2\,\bar{1})$. rr', po intersect in m, therefore m is $(3\,\bar{1}\,\bar{1})$. $mc_{,,}$, pp'' intersect in σ, therefore σ is $(4\,0\,\bar{3})$.

167. In a crystal of Calc Spar (fig. 69), p is $(1\,0\,0)$, p' is $(0\,1\,0)$, p'' is $(0\,0\,1)$, c is $(2\,\bar{1}\,\bar{1})$, and $c_{,,}$ is $(1\,1\,\bar{2})$. p, p' &c. (fig. 70) being the poles of p, p', &c. cp', pp'' intersect in r, therefore r is $(2\,0\,\bar{1})$. r'' is $(0\,2\,\bar{1})$. rr', cp intersect in m, therefore m is $(3\,\bar{1}\,\bar{1})$. Let y be $(h\,k\,l)$, then y' will be $(h\,l\,k)$. yy', $cc_{,,}$ intersect in $e_{,,}$, the pole of $(0\,1\,\bar{1})$.

m is in the zone-circle yy', therefore y is in the zone-circle $me_{,,}$, y is also in the zone-circle pr, therefore y is $(3\,0\,\bar{2})$. z, z' are in the zone-circle yy'. $zz' = 37^0.\ 8'$, therefore $mz = 18^0.\ 34'$. $\sin mo = \tan zm \cot zom$. Whence $\tan zom = \frac{1}{5}\sqrt{3}$, $\therefore \dfrac{h-l}{2h-k-l} = \dfrac{1}{5}$. But z is in the zone-circle my, the symbol of which is $[2\,3\,3]$, therefore (21) $2h + 3k + 3l = 0$. Whence $h = 15$, $k = -1$, $l = -9$, therefore z is $(15\,\bar{1}\,\bar{9})$.

168. To determine the positions of the following poles of a crystal of Calc Spar.

$n\ (2\,1\,1)$, $g\ (0\,1\,1)$, $f\ (\bar{1}\,1\,1)$, $m\ (3\,\bar{1}\,\bar{1})$, $l\ (\bar{1}\,3\,3)$, $\phi\ (\bar{2}\,3\,3)$, $d\ (\bar{5}\,3\,3)$, $h\ (\bar{4}\,5\,5)$, $v\ (3\,2\,0)$, $t\ (3\,1\,0)$, $\omega\ (4\,1\,0)$, $\lambda\ (4\,0\,\bar{1})$, $r\ (2\,0\,\bar{1})$, $y\ (3\,0\,\bar{2})$, $\sigma\ (4\,0\,\bar{3})$, $\theta\ (6\,0\,\bar{5})$, $z\ (15\,\bar{1}\,\bar{9})$, $x\ (2\,1\,\bar{2})$, $\delta\ (3\,1\,\bar{3})$, $b\ (7\,3\,\bar{5})$, $q\ (5\,3\,\bar{5})$.

Let p, p', p'' be the poles of $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$; o the pole of $(1\,1\,1)$, and let the poles in op' and the sector $coc_{,}$, op'' and the sector $c'oc_{,}$ be distinguished by one and two accents respectively. $pp'' = 74^0.\ 55'$, $pop'' = 120^0$. og' bisects pp'' and pop'', therefore, since $po = p''o$, $\sin \frac{1}{2} pp'' = \sin 60^0 \sin po$. Whence $po = 44^0.\ 36',6$. $\operatorname{Tan} go = -\frac{1}{2}\tan po$ (138), therefore $go = 26^0.\ 15'$. $\operatorname{Sin} \frac{1}{2} gg' = \sin 60^0 \sin go$, therefore $gg' = 45^0.\ 3'$. In like manner $no = 13^0.\ 52'$, $nn' = 23^0.\ 56'$; $fo = 63^0.\ 7'$, $ff' = 101^0.\ 9'$; $mo = 75^0.\ 47'$, $mm' = 114^0.\ 10'$; $lo = 38^0.17$, $ll' = 64^0.\ 53'$; $\phi o = 50^0.\ 58'$, $\phi\phi' = 84^0.\ 33'$; $do = 82^0.\ 47'$, $dd' = 118^0.\ 27'$; $ho = 55^0.\ 57'$, $hh' = 91^0.\ 42'$. $eo = 90^0$, $ec' = 90^0$, therefore $eg' = 90$, $pg' = 37^0.\ 27',5$, $pe = 52^0.\ 32',5$. Let $(u\,v\,w)$ be the symbol of any pole S in the zone-circle pp'', between e and g'. Therefore, substituting e, p, g' and their indices for p, q, r and their indices in (27), we have $\tan Se = \dfrac{u+w}{u-w}\tan pe$. Whence, since θ, σ, y, r, λ, ω', t', v' are in the zone-circle pp'', $\theta e = 6^0.\ 46'$, $\sigma e = 10^0.\ 34'$, $ye = 14^0.\ 38'$, $re = 23^0.\ 31'$, $\lambda e = 33^0.\ 8'$, $\omega' e = 65^0.\ 19'$, $t'e = 69^0.\ 2'$, $v'e = 81^0.\ 17'$, pe is known, therefore the distances of θ, σ, &c. from p may be found by subtraction.

$(\text{Tan } bo)^2 = \frac{112}{25} (\tan PO)^2$ (128), therefore $bo = 64^0. 24',5$.
$\tan bop = \frac{1}{2}\sqrt{3}$ (128), therefore $bop = 40^0. 53',6$, $\sin \frac{1}{2} bb' = \sin bop \sin bo$, therefore $bb' = 72^0. 22',5$, $\frac{1}{2} bob'' = 19^0. 6',4$, whence $bb'' = 34^0. 20'$. In like manner, $zo = 76^0. 32'$, $zop = 19^0. 6',4$, $zz' = 37^0. 8'$. q, x, δ, f'' are in the zone-circle ep', $ep' = 90^0$, $ef'' = 34^0. 25',5$. If $(u v w)$ be the symbol of any pole S in the zone-circle ep', then (27) $\tan Se = -\frac{v}{w} \tan ef''$. Whence $qe = 22^0. 21'$, $xe = 18^0. 55'$, $\delta e = 12^0. 52'$.

169. In a crystal of Tourmaline, the two ends of which are represented in figs. 71, 72, c, p, p', p'' are (1 1 1), (1 0 0), (0 1 0), (0 0 1) respectively. n'' is common to the zones pp', cp'', therefore n'' is (1 1 0). Similarly n' is (1 0 1), and n is (0 1 1). s is common to the zones pp'', nn'', therefore s is $(1\,0\,\bar{1})$. Similarly s'' is $(0\,1\,\bar{1})$, and s' is $(1\,\bar{1}\,0)$. l is common to the zones ss', cp, therefore l is $(2\,\bar{1}\,\bar{1})$. In fig. 72. p, $p_{,}$, $p_{,,}$ are respectively parallel to p, p', p'', therefore p is $(\bar{1}\,0\,0)$, $p_{,}$ $(0\,\bar{1}\,0)$, $p_{,,}$ $(0\,0\,\bar{1})$. g is common to the zones sp', $s'p''$, therefore g is $(1\,\bar{1}\,\bar{1})$. The faces parallel to c, l, g are wanting, therefore (124) c, l, g belong to hemihedral forms with inclined symmetric faces. Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{0\,\bar{1}\,1\}$, $\kappa\{2\,\bar{1}\,\bar{1}\}$, $\kappa\{0\,1\,1\}$, $\kappa\{1\,\bar{1}\,\bar{1}\}$.

170. p, m x, &c. (fig. 74), are the poles of the faces p, m, x, &c. of a crystal of Apatite (fig. 73). Let x, x', x'' be the poles of (1 0 0), (0 1 0), (0 0 1) respectively; and let p be the pole of (1 1 1). xx', px'' intersect in r_3, therefore r_3 is (1 1 0). In like manner r_2 is (1 0 1), and r_1 is (0 1 1). m is in px, and $pm = 90^0$, therefore (129) m is $(2\,\bar{1}\,\bar{1})$. m' is $(\bar{1}\,2\,\bar{1})$, m'' is $(\bar{1}\,\bar{1}\,2)$, and m_3 is $(1\,1\,\bar{2})$. mx', px'' intersect in x_3, therefore x_3 is $(2\,2\,\bar{1})$. mr_3, px' intersect in r', therefore r' is (1 1 4). xx'', mm' intersect in e, therefore e is $(1\,0\,\bar{1})$. ex', px'' intersect in z_3 therefore z_3 is $(1\,1\,\bar{1})$. $m'z_3$, px intersect in z, therefore z is $(5\,\bar{1}\,\bar{1})$. mx', xm_3 intersect in s, therefore s is $(4\,1\,\bar{2})$. xx', mx'' intersect in u', therefore

u' is $(2\,\bar{1}\,0)$. mz_3, xm_3 intersect in u, therefore u is $(5\,2\,\bar{4})$. pu, mm' intersect in c, therefore c is $(5\,\bar{4}\,\bar{1})$. The arcs $rr_{,}$, $xx_{,}$, $zz_{,}$, $uu_{,}$ are bisected in p, therefore (134) the forms to which the faces r, x, &c. belong are dirhombohedral. If twelve lunes be formed by zone-circles through p and each of the poles m, e, the poles c, $c_{,}$, u, $u_{,}$, in the alternate lunes are wanting, therefore (125) c, $c_{,}$ &c. belong to hemihedral forms with parallel faces. Hence the crystal (fig. 73) is a combination of the forms $\{1\,1\,1\}$, $\{2\,\bar{1}\,\bar{1}\}$, $\{0\,\bar{1}\,1\}$, $\{1\,0\,0\}$, $\{\bar{1}\,2\,2\}$, $\{0\,1\,1\}$, $\{4\,1\,1\}$, $\{\bar{1}\,1\,1\}$, $\{5\,\bar{1}\,\bar{1}\}$, $\{4\,1\,\bar{2}\}$, $\pi\{2\,\bar{1}\,0\}$, $\pi\{5\,2\,\bar{4}\}$, $\pi\{5\,\bar{4}\,\bar{1}\}$. It is cleavable parallel to the faces of the forms $\{1\,1\,1\}$, $\{2\,\bar{1}\,\bar{1}\}$. The faces u, s, x form an inverse zone.

The symbols of the other poles shewn in (fig. 74) are $a\ (5\,2\,\bar{1})$, $d\ (7\,1\,\bar{5})$, $f\ (3\,\bar{1}\,\bar{2})$, $b\ (2\,1\,\bar{2})$, $b'\ (8\,\bar{4}\,\bar{1})$.

The expressions in (128) give $\tan epm = \frac{1}{3}\sqrt{3}$, therefore the zone-circle $pasde$ makes an angle of 30^0 with pm. $\tan cpm = \frac{1}{2}\sqrt{3}$, therefore the zone-circle $c_{,}u_{,}puc$ makes an angle of $40^0.\,53',6$ with pm. $\tan bpm = \frac{3}{5}\sqrt{3}$, therefore the zone-circle $b_{,}pb$ makes an angle of $46^0.\,6'$ with pm. If $xp = D$, we have from (128), $\tan D = 2\tan rp = \frac{1}{2}\tan zp = \dfrac{2}{\sqrt{3}}\tan ap$

$$= \frac{1}{\sqrt{3}}\tan sp = \frac{1}{2\sqrt{3}}\tan dp = \frac{1}{\sqrt{7}}\tan up = \frac{1}{\sqrt{13}}\tan bp.$$

$$mpm_3 = 60^0, \ \therefore\ epm_3 = 30^0,\ upm_3 = 19^0.\,6',4,\ bpm_3 = 13^0.\,4'.$$

$$\text{Sin}\, ap = \cos a'm_3e' = \tan 60^0 \cot xm_3 = \tan 30^0 \cot sm_3$$
$$= \tan 19^0.6',4 \cot um_3 = \tan 13^0.\,4' \cot bm_3 = \tan 40^0.\,53',6 \cot um.$$

$$\text{Sin}\, sp = \cos s'm_3e' = \tan 60^0 \cot zm_3 = \tan 30^0 \cot dm_3.$$

$$\text{Cos}\, rm_3 = \cos 60^0 \sin rp,\ \ \cos am_3 = \cos 30^0 \sin ap.$$

171. r, p, z, &c. (fig. 76) are the poles of the faces r, p, z, &c. of a crystal of Quartz (fig. 75). The distance between any two adjacent poles r, r_2, r_3, &c. is 60^0. The arcs pr, $p'r'$, $p''r''$ are each $38^0.\,13'$, and are perpendicular to rr_2, therefore they pass through o, the pole of rr_2. There-

fore, if p be $(1\,0\,0)$, p' $(0\,1\,0)$, p'' $(0\,0\,1)$, o will be $(1\,1\,1)$, r $(2\,\bar{1}\,\bar{1})$, r' $(\bar{1}\,2\,\bar{1})$, r'' $(\bar{1}\,\bar{1}\,2)$. rp', $r'p$ intersect in z'', therefore z'' is $(2\,2\,\bar{1})$. rp, $r''p$ intersect in s, therefore s is $(4\,\bar{2}\,1)$. $ar = 18^0.\,11'$, therefore $\tan ao = 4 \tan po$, therefore (138) a is $(3\,\bar{1}\,\bar{1})$. $a_{,}r_{,} = ar$, therefore $a_{,}o = ao$, therefore (133) $a_{,}$ is $(\bar{7}\,5\,5)$. n, x are in the zone-circle pr_2. $xr_2 = 18^0.\,29'$, $nr_2 = 12^0$. If $(u\,v\,w)$ be the symbol of any pole S in pr_2, substituting r_2, p, z'' and their indices for P, Q, R, and their indices in (27), and observing that $\cot z''r_2 = -\cot pr_2$, we have $(2u - 5v) \tan Sr_2 = (2u + v) \tan pr_2$. The symbol of the zone-circle pr_2 is $[0\,1\,2]$, $\therefore\ v + 2w = 0$, $\tan pr_2 = 7 \tan xr_2$, $\therefore\ x$ is $(2\,\bar{2}\,1)$, $\tan pr_2 = 13 \tan nr_2$, $\therefore\ n$ is $(8\,\overline{10}\,5)$. pz, $aa_{,}$ are bisected in o, therefore (134) p, a belong to dirhombohedral combinations. There are no poles s, x, n in the alternate lunes formed by zone-circles through o and each of the poles r, therefore (126) s, x, n belong to hemihedral forms with inclined asymmetric faces. Hence the crystal (fig. 75) is a combination of the forms $\{2\,\bar{1}\,\bar{1}\}$, $\{1\,0\,0\}$, $\{\bar{1}\,2\,2\}$, $\{3\,\bar{1}\,\bar{1}\}$, $\{\bar{7}\,5\,5\}$, $\{4\,\bar{2}\,1\}$, $a\{2\,\bar{2}\,1\}$, $a\{8\,\overline{10}\,5\}$. The zone formed by the faces p, s, x, n, r_2 is direct.

The symbols of the other poles shewn in (fig. 76) are b $(13\,\bar{2}\,\bar{2})$, $b_{,}$ $(\bar{7}\,8\,8)$, m $(7\,\bar{2}\,\bar{2})$, $m_{,}$ $(\bar{5}\,4\,4)$, e $(16\,\bar{5}\,\bar{5})$, $e_{,}$ $(\bar{4}\,3\,3)$, c $(5\,\bar{2}\,\bar{2})$, $c_{,}$ $(\overline{13}\,8\,8)$, w $(5\,\bar{4}\,2)$, y $(7\,\bar{8}\,4)$, t $(3\,\bar{4}\,2)$.

CHAPTER V.

PRISMATIC SYSTEM.

172. In the prismatic system the axes make right angles with each other.

173. The holohedral form $\{h\,k\,l\}$ is bounded by all the faces having for their symbols the different combinations of $\pm h$, $\pm k$, $\pm l$, each index having always the same place. When h, k, l are all finite, the form $\{h\,k\,l\}$ will have the eight faces

$$\begin{array}{cccc} h\,k\,l & h\,\bar{k}\,\bar{l} & \bar{h}\,k\,\bar{l} & \bar{h}\,\bar{k}\,l \\ \bar{h}\,\bar{k}\,\bar{l} & \bar{h}\,k\,l & h\,\bar{k}\,l & h\,k\,\bar{l} \end{array}$$

When one of the indices is zero, the number of faces will be four. When two of the indices are zero, the number of faces will be two.

The arrangement of the poles of $\{h\,k\,l\}$ on the surface of the sphere of projection is shewn in fig. 77.

174. The form bounded by all the faces of $\{h\,k\,l\}$ which have either an odd number of positive indices, or an odd number of negative indices, is said to be hemihedral with inclined faces, and will be denoted by $\kappa\{h\,k\,l\}$, where $h\,k\,l$ is the symbol of any one of its faces. The forms are said to be direct or inverse according as they have an odd number of positive or of negative indices, and their symbols are contained respectively in the upper and lower lines of the table above.

If the surface of the sphere of projection be divided into eight triangles by zone-circles through every two of the poles

of (1 0 0), (0 1 0), (0 0 1), the poles of the direct form will be found in four alternate triangles, one of which contains (1 1 1), and those of the inverse form in the four remaining triangles.

175. The form bounded by all the faces of $\{h\,k\,l\}$, in the symbols of which the sign of one of the indices remains unchanged, is said to be hemihedral with symmetric faces, and may be denoted by prefixing to $\{h\,k\,l\}$, where $(h\,k\,l\}$ is the symbol of one of its faces, σ_1, σ_2 or σ_3, according as the first, second or third index preserves its sign unchanged, and may be called direct or inverse according as that index is positive or negative.

The poles of the hemihedral form with symmetric faces will be found in one of the hemispheres into which the surface of the sphere is divided by a zone-circle through two of the three poles (1 0 0), (0 1 0), (0 0 1).

176. To determine the position of any pole.

Let the axes of the crystal meet the surface of the sphere of projection in X, Y, Z (fig. 78). Let a, b, c be the parameters of the crystal. P the pole of $(h\,k\,l)$.

Since the axes of the crystal are rectangular, YZ, ZX, XY are quadrants, $\therefore$ $\cos YZ = 0$, $\cos ZX = 0$, $\cos XY = 0$, therefore X, Y, Z are the poles of (1 0 0), (0 1 0), (0 0 1), and the angles at X, Y, Z right angles.

$$\cos PX = \sin PY \cos PYX = \sin PZ \cos PZX,$$

$$\cos PY = \sin PZ \cos PZY = \sin PX \cos PXY,$$

$$\cos PZ = \sin PX \cos PXZ = \sin PY \cos PYZ.$$

$$\cot PX = \tan PZY \cos PXY = \tan PYZ \cos PXZ,$$

$$\cot PY = \tan PXZ \cos PYZ = \tan PZX \cos PYX,$$

$$\cot PZ = \tan PYX \cos PZX = \tan PXY \cos PZY.$$

But

$$\frac{a}{h}\cos PX = \frac{b}{k}\cos PY = \frac{c}{l}\cos PZ.$$

Whence

$$\tan PXY = \frac{l}{k}\frac{b}{c}, \quad \tan PYZ = \frac{h}{l}\frac{c}{a}, \quad \tan PZX = \frac{k}{h}\frac{a}{b}.$$

$$\cot PX = \frac{h}{k}\frac{b}{a}\cos PXY = \frac{h}{l}\frac{c}{a}\cos PXZ,$$

$$\cot PY = \frac{k}{l}\frac{c}{b}\cos PYZ = \frac{k}{h}\frac{a}{b}\cos PYX,$$

$$\cot PZ = \frac{l}{h}\frac{a}{c}\cos PZX = \frac{l}{k}\frac{b}{c}\cos PZY.$$

177. It appears from (176), that the distance of any pole of $\{h\,k\,l\}$ from the three nearest poles of $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,0\,1\}$ are respectively equal to the distances of any other pole of $\{h\,k\,l\}$ from the three nearest poles of $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,0\,1\}$. Hence the poles of $\{h\,k\,l\}$ are symmetrically arranged with respect to each of the three zone-circles through every two of the poles of the forms $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,0\,1\}$.

178. The arrangement of the poles of $\{h\,k\,l\}$, and of $\sigma\{h\,k\,l\}$, will be symmetrical in any two adjacent triangles formed by zone-circles through every two of the poles of $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,0\,1\}$, and similar in any two alternate triangles. The arrangement of the poles of $\kappa\{h\,k\,l\}$ will be similar in each of the triangles in which they occur.

179. In either of the hemihedral forms, the poles of the direct and inverse forms may be made to change places with each other, by causing the sphere of projection to revolve through two right angles round the poles of one of the forms $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,0\,1\}$.

180. In the form $\{0\,k\,l\}$, if L be the distance between two poles differing only in the sign of l,

$$\tan \tfrac{1}{2}L = \frac{l}{k}\frac{b}{c}.$$

181. In the form $\{h\,0\,l\}$, if L be the distance between two poles differing only in the sign of l,

$$\tan \tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}.$$

182. In the form $\{h\,k\,0\}$, if H be the distance between two poles differing only in the sign of h,

$$\tan \tfrac{1}{2}H = \frac{h}{k}\frac{b}{a}.$$

183. In the form $\{h\,k\,l\}$, if H, K, L be the distances between any two poles, the symbols of which differ only in the signs of h, k, l respectively,

$$\text{if}\quad \tan\phi = \frac{k}{h}\frac{a}{b},\quad \tan \tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}\cos\phi,$$

$$\sin \tfrac{1}{2}K = \cos \tfrac{1}{2}L \sin\phi,\quad \sin \tfrac{1}{2}H = \cos \tfrac{1}{2}L \cos\phi.$$

184. Let P be the pole of $\{h\,k\,l\}$, Q the pole of $\{p\,q\,r\}$. Then as in (97), when Q is in the zone-circle PX,

$$\frac{h}{p}\frac{\tan PX}{\tan QX} = \frac{k}{q} = \frac{l}{r}.$$

When Q is in the zone-circle PY,

$$\frac{k}{q}\frac{\tan PY}{\tan QY} = \frac{l}{r} = \frac{h}{q}.$$

When Q is in the zone-circle PZ,

$$\frac{l}{r}\frac{\tan PZ}{\tan QZ} = \frac{h}{p} = \frac{k}{q}.$$

185. To find the distance between any two poles.

Let P, Q (fig. 78) be the poles of $(h\,k\,l)$, $(p\,q\,r)$; X, Y, Z poles of the forms $\{0\,0\,1\}$, $\{0\,1\,0\}$, $\{1\,0\,0\}$. Let PQ meet the zone-circle of which Z is a pole in M. Then the tangents of MZX, PZX, QZX can be found in terms of h, k, l, p, q, r and of two of the parameters a, b, c; therefore PZM, QZM are known. PZ can be found in terms of h, k, l, and of the parameters a, b, c. MZ is a quadrant, therefore

$$\cos PM = \sin PZ \cos PZM,$$

$$\frac{\tan QM}{\tan PM} = \frac{\tan QZM}{\tan PZM}.$$

Therefore, PM, QM being known, PQ which is their sum or difference is known.

186. If the distance between any two poles of either of the forms $\{0\,k\,l\}$, $\{h\,0\,l\}$, $\{h\,k\,0\}$ be given, the ratio of the indices may be obtained from the expressions in (180)...(182).

187. In the form $\{h\,k\,l\}$, the distances between any pole and each of two others, or their supplements, will be two of the arcs H, K, L; therefore two of the arcs H, K, L being known, ϕ and thence the ratios of h, k, l may be found from (182).

188. The ratios of the parameters may be found from the expressions in (180)...(182), having given the distances between the poles of two of the forms $\{0\,k\,l\}$, $\{h\,0\,l\}$, $\{h\,k\,0\}$; or from the expressions in (183), having given the distance between any pole of $\{h\,k\,l\}$, and each of two others not all in the same zone-circle.

189. The ratios of the parameters may also be found from the distances between three given poles in one zone-circle.

Let P, Q, R (fig. 79) be the three poles. Let PR meet YZ, ZX, XY in L, M, N respectively. Then, the symbols of P, Q, R being known, the symbols of the poles L, M, N

may be found by (17). Therefore, PL, PM, PN may be found by (27). Therefore the distances between L, M, N are known.

$$\frac{\tan LY}{\tan LZ} = \frac{\tan NL}{\tan LM}, \quad \frac{\tan MZ}{\tan MX} = \frac{\tan LM}{\tan MN}, \quad \frac{\tan NX}{\tan NY} = \frac{\tan MN}{\tan NL}.$$

Hence, the places and symbols of L, M, N being known, the ratios of a, b, c may be found by (180)...(182).

190. To determine the figure and angles of the form $\{h\,k\,l\}$, when h, k, l take particular values.

The angle between normals to any two faces of the same form is obtained from the expressions in (180)...(183), and will be denoted by the letter which, in the accompanying figure, is placed upon the edge formed by their intersection. The arrangement of the poles is shewn in fig. 77. The number of faces is given in (173).

191. The three forms $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,0\,1\}$, have each two parallel faces, the faces of any one form being perpendicular to those of each of the other two.

Either of these forms may become hemihedral according to the second law.

192. The form $\{0\,k\,l\}$ (fig. 80), has four faces perpendicular to the faces of $\{1\,0\,0\}$.

$$\text{Tan}\,\tfrac{1}{2}L = \frac{l}{k}\frac{b}{c}, \quad K = 180^0 - L.$$

193. The form $\{h\,0\,l\}$ (fig. 81), has four faces perpendicular to the faces of $\{0\,1\,0\}$.

$$\text{Tan}\,\tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}, \quad H = 180^0 - L.$$

194. The form $\{h\,k\,0\}$ (fig. 82) has four faces perpendicular to the faces of $\{0\,0\,1\}$.

$$\text{Tan}\,\tfrac{1}{2}H = \frac{h}{k}\frac{b}{a}, \quad K = 180^0 - H.$$

195. Either of the preceding forms may become hemihedral with symmetric faces. The half-form will consist of any two adjacent faces.

196. The form $\{h\,k\,l\}$ (fig. 83) has eight faces.

$$\text{If}\quad \tan\phi = \frac{k}{h}\frac{a}{b}, \quad \tan\tfrac{1}{2}L = \frac{l}{h}\frac{a}{c}\cos\phi,$$

$$\sin\tfrac{1}{2}K = \cos\tfrac{1}{2}L\sin\phi, \quad \sin\tfrac{1}{2}H = \cos\tfrac{1}{2}L\cos\phi.$$

197. The hemihedral form with inclined faces $\kappa\{h\,k\,l\}$ is an irregular tetrahedron, the edges of which are parallel to the faces (1 0 0), (0 1 0), (0 0 1). If normals to the faces that meet in these edges make with each other the angles T, V, W respectively,

$$T = 180^0 - H, \quad V = 180^0 - K, \quad W = 180^0 - L.$$

198. The hemihedral form with symmetric faces consists of four faces which make one of the solid angles of fig. 83.

Examples.

199. Let m, k, p, &c. (fig. 85), be the poles of the faces m, k, p, &c. of a crystal of Aragonite (fig. 84). It is found that the zone-circles mm', kk' intersect at right angles in h, and that the poles of the faces are symmetrically arranged with respect to each of the circles mm', kk' and a circle YZ which intersects mm', kk' at right angles in Y, Z. Let h be (1 0 0), k (1 0 1), m (1 1 0). Then Y, Z will be (0 1 0), (0 0 1) respectively. It appears from the symmetrical arrangement of the poles with respect to the great circles mm', kk', YZ, that pp''', pp'' pass through Y, Z respectively. k, m are in pp''', pp'' respectively. Hence p is the intersection of Yk, Zm, therefore p is (1 1 1). s is the intersection of hp, mk, therefore s is (2 1 1). s'' is $(\bar{2}\,\bar{1}\,1)$, s''' is $(2\,\bar{1}\,1)$. i is the intersection of hk, ss''', therefore i is (2 0 1). n is the intersection of pk, ss'', therefore n is (2 1 2). x is the intersection of mn, hk, therefore x is (1 0 2). Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{1\,0\,1\}$, $\{2\,0\,1\}$, $\{1\,0\,2\}$, $\{1\,1\,0\}$, $\{1\,1\,1\}$, $\{2\,1\,1\}$, $\{2\,1\,2\}$. The crystal is cleavable parallel to the faces of $\{1\,0\,0\}$. $\{1\,0\,1\}$, $\{1\,1\,0\}$.

It is found that $mm' = 63^0.\,50'$, $kk' = 71^0.\,34'$, very nearly. Therefore $mh = 58^0.\,5'$, $kh = 54^0.\,13'$. $\tan kh = 2\tan ih = \frac{1}{2}\tan xh$ (183). Hence $ih = 34^0.\,45'$, $xh = 70^0.\,11'$. $kZ = 35^0.\,47'$, $\cos pZk = \tan kZ \cot pZ$, $\sin kZ = \cot pZk \tan pk$. Whence $pZ = 53^0.\,44',5$, $pk = 43^0.\,11',5$, therefore $pp'' = 107^0.\,29'$, $pp''' = 86^0.\,23'$. $\cos ph = \cos mh \cos pm$, therefore $ph = 64^0.\,46'$, $pp' = 50^0.\,28'$. $\tan nY = 2\tan pY$, $\tan ph = 2\tan sh$ (183), therefore $nY = 64^0.\,51'$, $nk = 25^0.\,9'$, $sh = 46^0.\,42'$. $\tan pZh = 2\tan sZh$ (175), therefore $sZh = 38^0.\,45',5$. $\sin iZ = \tan si \cot sZh$, $\cos sZh = \tan iZ \cot sZ$, therefore $si = 33^0.\,24',5$, $sZ = 61^0.\,35'$. $\tan sZ = 2\tan nZ$ (183), therefore $nZ = 42^0.\,45'$, $\cos nh = \cos nk \cos kh$, therefore $nh = 58^0.\,2',5$.

Let a, b, c be the parameters of the crystal. Then, since p is the pole of (1 1 1), and h the pole of (1 0 0), $a\cos ph = b\cos pY = c\cos pZ$. Therefore a, b, c are proportional to

sec ph, sec pY, sec pZ, or to the numbers 2,3457, 1,4610, 1,6908 respectively.

If we change the parameters by the rule in (29) so that s may be (1 1 1), the symbols of the other faces will be h (1 0 0), m (1 2 0), i (1 0 1), k (1 0 2), x (1 0 4), p (1 2 2) n (1 1 2). The new parameters will be proportional to the numbers 1,1728, 1,4610, 1,6908.

200. In a crystal of Sulphate of Magnesia with seven proportionals of water (fig. 86), the zones which serve to determine the symbols of the faces are $nlte$, $vlsp$, mll'', nsm, vtm. e is (1 0 0), p (0 1 0), n (0 1 1), v (1 0 1). Therefore (17) the symbols of the remaining faces are l (1 1 1), l'' ($\bar{1}\,\bar{1}$ 1), m (1 1 0), t (2 1 1), s (1 2 1), q (0 2 1), r (2 0 1). The forms to which s, t belong want the faces which have their poles in the alternate octants formed by the zone-circles through every two of the poles (1 0 0), (0 1 0), (0 0 1), therefore (173) they are hemihedral with inclined faces. Hence the crystal is a combination of the forms $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,1\,1\}$, $\{1\,0\,1\}$, $\{1\,1\,0\}$, $\{1\,1\,1\}$, $\{0\,2\,1\}$, $\{2\,0\,1\}$, $\kappa\{2\,1\,1\}$, $\kappa\{1\,2\,1\}$. It is cleavable parallel to the faces of $\{1\,0\,0\}$.

The form to which the faces l belong is frequently hemihedral with inclined faces.

If e, m, l be the poles of the faces e, m, l, $em = 45^0.\ 15'$, $ml = 51^0$.

201. In a crystal of Topaz (fig. 87), the zones which serve to determine the symbols of the faces are $ulmm'$, $ynpn'y'$, $mosps''o''m''$, $m'o's'ps'''o'''m'''$, $mnx'''o'''m''$, $m'oxnm'''$, $uon'u''$, $uo'y'$, $lxpx''l''$, $xss'x'$. p is perpendicular to the faces of the zone mm', therefore if p be (0 0 1), the faces (1 0 0), (0 1 0) will belong to the zone mm'. Let o be (1 1 1), o' ($\bar{1}$ 1 1). Then o'' will be ($\bar{1}\,\bar{1}$ 1), o''' (1 $\bar{1}$ 1). m, m' will be (1 1 0), ($\bar{1}$ 1 0) respectively. n will be (2 0 1), u (3 1 0), y' ($\bar{4}$ 0 1). If m, m', l, l' be the poles of the faces m, m', l, l', it is found that $\tan\frac{1}{2}ll' = 2\tan\frac{1}{2}mm'$, therefore l is (2 1 0). Hence x is (4 2 3), x' ($\bar{4}$ 2 3), s (2 2 3). Hence the crystal

is a combination of the forms $\{0\,0\,1\}$, $\{1\,1\,0\}$, $\{2\,1\,0\}$, $\{3\,1\,0\}$, $\{2\,0\,1\}$, $\{4\,0\,1\}$, $\{1\,1\,1\}$, $\{2\,2\,3\}$, $\{4\,2\,3\}$. The crystal is cleavable parallel to the faces of $\{0\,0\,1\}$, $\{2\,0\,1\}$, $\{0\,2\,1\}$.

The forms of Topaz are sometimes hemihedral with symmetric faces (174). Thus the forms to which o, x, p, t (t is common to the zones oo''', nn', its symbol is (101)) occasionally wanting the faces on one side of the zone mm'; and the form of which i is a face (i is common to the zones mo', $m'o$, its symbol is (021)), has been observed to want the faces on one side of the zone nn'.

If m, u, p, &c. be the poles of the faces m, u, p, &c., $mm' = 55^0.\,41'$, $ll' = 93^0.\,8'$, $uu' = 115^0.\,29'$, $pn = 43^0.\,30',5$, $py = 62^0.\,13'$, $po = 45^0.\,27',5$, $ps = 34^0.\,7'$, $px = 41^0.\,4'$.

CHAPTER VI.

OBLIQUE ~~PRISMATIC~~ SYSTEM.

202. In the oblique prismatic system one axis OY, is perpendicular to each of the other two axes.

203. The holohedral form $\{h\,k\,l\}$ is bounded by all the faces which have for their symbols the different combinations of $\pm h$, $\pm k$, $\pm l$, in which each of the three indices has always the same place, and the first and second indices are taken with the same signs. When k is finite the form has the four faces

$$h\,k\,l \qquad \bar{h}\,k\,\bar{l} \qquad h\,\bar{k}\,l \qquad \bar{h}\,\bar{k}\,\bar{l}$$

When k is zero, or when each of the other two indices are zero the number of faces will be two.

204. The hemihedral form which we shall denote by $\sigma\{h\,k\,l\}$, where $(h\,k\,l)$ is the symbol of one of its faces, is bounded by the faces of $\{h\,k\,l\}$ in the symbols of which k has the same sign.

The poles of the two half-forms lie on different sides of the zone-circle [1 0 0, 0 0 1].

205. To determine the position of any pole.

Let the axes of the crystal meet the surface of the sphere 88
of projection in X, Y, Z (fig. 87). Let C be the pole (0 0 1), A the pole of (1 0 0), P the pole of $(h\,k\,l)$.

The axis OY is perpendicular to each of the other two axes, therefore XY, YZ are quadrants; therefore $\cos XY = 0$, $\cos ZZ = 0$. Therefore Y is the pole of (0 1 0). $\cos CX = 0$,

$\cos CY = 0$, $\cos AZ = 0$, $\cos AY = 0$. Therefore CX, CY, AZ, AY are quadrants. Therefore C, A are in the great circle ZX, and $CA + ZX = 180^0$.

$$\cos PX = \sin PY \cos PYX = \sin PY \sin PYC,$$

$$\cos PZ = \sin PY \cos PYZ = \sin PY \sin PYA.$$

But

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ;$$

$$\therefore \quad \frac{a}{h} \sin PYC = \frac{c}{l} \sin PYA, \quad \therefore \text{ if } \tan\theta = \frac{h}{l}\frac{c}{a},$$

$$\tan\tfrac{1}{2}(PYC - PYA) = \tan\tfrac{1}{2}CA \tan(45^0 - \theta).$$

$$\cot PY = \frac{k}{h}\frac{a}{b} \sin PYC = \frac{k}{l}\frac{c}{b} \sin PYA.$$

$$\cos PA = \sin PY \cos PYA,$$

$$\cos PC = \sin PY \cos PYC.$$

206. The arc joining two poles of $\{h\,k\,l\}$, the symbols of which differ only in the sign of k, is manifestly bisected at right angles by the zone-circle $[0\,0\,1,\ 1\,0\,0]$. Hence, if the surface of the sphere of projection be divided into two hemispheres by the great circle $[0\,0\,1,\ 1\,0\,0]$, the arrangement of the poles of $\{h\,k\,l\}$ in the two hemispheres will be symmetrical.

The arc joining the two poles of $\kappa\{h\,k\,l\}$ will be bisected in a pole of the form $\{0\,1\,0\}$.

207. Let P be the pole of $(h\,k\,l)$, Q the pole of $(p\,q\,r)$, and let Q be in the zone-circle PY. Then $PYC = QYC$, $PYA = QYA$, therefore (205).

$$\frac{k \tan PY}{q \tan QY} = \frac{l}{r} = \frac{h}{p}.$$

208. To find the distance between any two poles.

Let P, Q (fig. 89) be the poles of $(h\,k\,l)$, $(p\,q\,r)$, A, Y, C the poles of $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$. Let PQ meet CA in M. Then the symbol of M may be found, and MYA, PYA, QYA, PY may be found in terms of h, k, l, p, q, r, a, c and the angle between the axes. MY is a quadrant, therefore

$$\cos PM = \cos PYM \sin PY,$$

$$\frac{\tan QM}{\tan PM} = \frac{\tan QYM}{\tan PYM},$$

whence, PM, QM being known, PQ is known.

209. Having given the distances of the pole of $(h\,k\,l)$ from the poles of $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$; to find ZX, and the ratios of a, b, c.

Let P, A, Y, C (fig. 88) be the poles of $(h\,k\,l)$, $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$. Then (205)

$$\cos PA = \sin PY \cos PYA,$$

$$\cos PC = \sin PY \cos PYC.$$

Whence PYA, PYC, and, therefore, AC and ZX become known. The ratio of a to c is given by the equation

$$\frac{a}{h} \sin PYC = \frac{c}{l} \sin PYA;$$

and the ratio of b to a or c by the equations

$$\cot PY = \frac{k}{h}\frac{a}{b} \sin PYC = \frac{k}{l}\frac{c}{b} \sin PYA.$$

210. P, Q, R (fig. 90) are three poles in the zone-circle CA, C, A being the poles of $(0\,0\,1)$, $(1\,0\,0)$; T, T' are two poles of the same form. Having given PQ, QR, TT', and the symbols of P, Q, R', T, to find the inclination of the axes and the ratios of the parameters.

Let TT' meet PQR in S. P, Q, R, S, A, C are in the same zone-circle, and their symbols are known, therefore (26),

the distances of S, C, A from P and R may be found. Whence, CA being known, the inclination of the axes is known. Y bisects TT', therefore, TY, CS, AS being known, TC, TA may be found, and the ratios of the parameters determined as in (209).

211. M, M' (fig. 91) are the poles of two faces of any form equidistant from Y; N, N' are the poles of two faces of any other form, also equidistant from Y. Having given MM', NN', MN; to find the inclination of the axes, and the ratios of the parameters.

Let MM', NN', MN meet CA in P, Q, R, C, A being the poles of (0 0 1), (1 0 0). The symbols of M, N being known, those of P, Q, R may be found. MN, YM, YN being known, PQ, which measures the angle MYN, may be found.

$$\sin PR = \cot R \cot YM, \quad \sin QR = \cot R \cot YN,$$

$$\therefore \frac{\tan \frac{1}{2}(PR - QR)}{\tan \frac{1}{2}(PR + QR)} = \frac{\sin (NY - MY)}{\sin (NY + MY)}.$$

Whence, knowing PQ, PR, QR are found. PQ, QR being known, the places of C, A, and the ratios of the parameters may be found by the methods of (209) and (210).

212. P, Q, R (fig. 92) are three poles in the same zone-circle. T, T' two poles of the same form equidistant from Y. Having given PQ, QR, TT', and the symbols of P, Q, R, T; to find the elements of the crystal.

Let PR meet ZX in M and TY in S; and let TY, RY, PY meet ZX in s, r, p. Then the symbols of M, S, p, r, s can be found. PQ, PR and the symbols of M, P, Q, R, S, are known, therefore NP, MR, SP, SR may be found by (26). RY, RM determine RMY. RMY, PM, RM, SM determine pM, rM, sM. Whence, knowing the symbols of p, r, s, the places of C, A can be found by (210), the elements of the crystal may then be found by the methods already given.

213. To find the indices of any face when referred to the axes of the zones $[e\,0\,g,\ 0\,1\,0]$, $[0\,0\,1,\ 1\,0\,0]$, $[p\,0\,r,\ 0\,1\,0]$ as crystallographic axes.

The symbols of the three zones are $[\bar{r}\,0\,p]$, $[0\,1\,0]$, $[g\,0\,\bar{e}]$, therefore (28)

$$\mathrm{e} = -r,\ \mathrm{f} = 0,\ \mathrm{g} = p,\ \mathrm{h} = 0,\ \mathrm{k} = 1,\ \mathrm{l} = 0,\ \mathrm{p} = g,\ \mathrm{q} = 0,\ \mathrm{r} = -e.$$

Hence, if u, v, w be the indices of any face when referred to the old axes, u', v', w' its indices when referred to the new axes,

$$u' = pw - ru,\quad v' = v,\quad w' = gu - ew.$$

214. The form $\{0\,1\,0\}$ has two parallel faces.

215. The form $\{h\,0\,l\}$ has two faces parallel to each other, and perpendicular to the faces of $\{0\,1\,0\}$.

216. The form $\{h\,k\,l\}$ has four faces. Normals to two faces adjacent to $(0\,1\,0)$ make with each other an angle $180^0 - K$, where $90^0 - \frac{1}{2}K = PY$.

217. The hemihedral form $\sigma\{h\,k\,l\}$ has two faces, normals to which make with each other an angle $180^0 - K$.

Examples.

218. In a crystal of Epidote (fig. 93), the zones which determine the symbols of the faces are $metlrm'$, $mkoo'k'm'$, $tuzz'u't'$, $lyqq'y'l'$, $lnz'l'$, $rnn'r'$, $mdzqnxm'$, $muym'$, $ryzox'r'$, $tyno'd't'$, $edd'e'$. Let m be $(1\,0\,0)$, l $(0\,0\,1)$, n $(1\,1\,1)$, n' $(\bar{1}\,1\,1)$. Therefore (17) the symbols of the faces will be r $(\bar{1}\,0\,1)$, q $(0\,1\,1)$, z $(1\,1\,1)$, t $(1\,0\,1)$, o $(2\,1\,0)$, y $(0\,1\,2)$, x $(\bar{3}\,1\,1)$, i $(\bar{1}\,0\,3)$, u $(2\,1\,2)$, k $(4\,1\,0)$, d $(3\,1\,1)$, e $(3\,0\,1)$. In some crystals a face f has been observed common to the zones mt, un'; a face s common to the zones mt, ky, and a face b common to the zones mo, lg. Therefore f is $(1\,0\,3)$, s is $(2\,0\,1)$, and b is $(0\,1\,0)$. The crystal is cleavable parallel to the faces m, t.

Let m, l, r, &c. (fig. 94), be the poles of the faces m, l, r, &c. Having given $rt = 51^0.\,41'$, $tm = 64^0.\,36'$, $nn' = 70^0.\,33'$; to find the positions of the remaining poles.

Let $(u\,v\,w)$ be the symbol of any pole S in the zone-circle rtm; then, substituting r, t, m and their indices for P, Q, R and their indices in (27),

$$\frac{\tan rS - \tan rm}{\tan rt - \tan rm} = \frac{2w}{u + w},$$

$$\therefore\ (u + w) \tan rS = 2w \tan rt + (u - w) \tan rm.$$

Whence $lr = 25^0.\,44',5$; $fr = 34^0.\,55'5$; $er = 81^0.\,34$; $ir = 29^0.\,21',5$; $sr = 18^0.\,6'$. If $nml = \phi$, $\tan nr = \tan\phi \sin mr$, $\tan iv = \tan\phi \sin mi$. Whence $iv = 41^0.\,39',5$, $vv' = 96^0.\,41'$. In like manner $qq = 64^0.\,46'$, $zz' = 70^0.\,9'$, $dd' = 96^0.\,10'$. If $ntr = \psi$, $\tan nr = \tan\psi \sin tr$, $\tan om = \tan\psi \sin tm$. Whence $om = 58^0.\,26'$, $oo' = 63^0.\,8'$. $\operatorname{Tan} by = 2 \tan bq$ (207), therefore $by = 51^0.\,45'$, $yy' = 103^0.\,30'$. $\operatorname{Tan} bu = 2 \tan bz$, $\tan bk = 2 \tan bo$. Whence $bu = 54^0.\,33'$, $uu' = 109^0.\,6'$, $bk = 50^0.\,51',5$, $kk' = 101^0.\,43'$.

If the zones which intersect mt in f, s, had not been found, the distances of f, s from the poles of some known face in mt would have been requisite in order to determine their symbols.

Suppose tf, ts had been measured, and that we had found $tf = 19^0.\ 37'$, $ts = 69^0.\ 47'$. We have $tr = 51^0.\ 41'$, $rm' = 63^0.\ 43'$. Therefore, substituting t, r, m' and their indices for P, Q, R and their indices, and f for S in (27), if $(u\ v\ w)$ be the symbol of f, $u = 1$, $v = 0$, $w = 3$. Therefore f is (103). In like manner s will be $(\bar{2}\ 0\ 1)$.

To find the inclination of the axes OZ, OX, and the parameters.

Let OZ, OX meet the surface of the sphere of projection in Z, X. Then, since l is (0 0 1) and m is (1 0 0), $mZ = 90^0$, $lX = 90^0$. $ml = 90^0.\ 32',5$, therefore $ZX = 89^0.\ 27',5$. The axis OY meets the surface of the sphere of projection in b. If the parameters of the crystal be a, b, c, since u is (212), $\frac{1}{2}a \cos uX = b \cos ub = \frac{1}{2}c \cos uZ$. $\sec uX = \sec ut \operatorname{cosec} tl$, $\sec uZ = \sec ut \operatorname{cosec} tm$. Therefore a, b, c are proportional to $2 \sec ut \operatorname{cosec} tl$, $\sec ub$, $\sec ut \operatorname{cosec} tm$.

To find the symbols of the faces referred to the axes of the zones zz', mt, oo' as crystallographic axes. The symbols of the zones zz', mt, oo', when referred to the old axes, are $[\bar{1}\ 0\ 1]$, $[0\ 1\ 0]$, $[0\ 0\ 1]$ respectively. Therefore (28), if $(u\ v\ w)$ be the symbol of any face referred to the old axes, $(u'\ v'\ w')$ its symbol when referred to the new axes, $u' = u - w$, $v' = v$, $w' = w$. If the new axes OZ', OX' meet the surface of the sphere of projection in Z', X', $Z'X' = 180^0 - mt = 115^0.\ 24'$.

219. In a crystal of Felspar (fig. 95), the zones which determine the symbols of the faces are $tzmz't'$, $pqxyp$, $pnmn'p'$, $xomox$, $qoznq$, $potp$, $yonty$, m is perpendicular to the faces p, q, x, y, therefore m is (0 1 0). Let t be (1 1 0), t' $(\bar{1}\ 1\ 0)$, o (1 1 1). Then (17) the symbols of the remaining faces will be p (0 0 1), n (0 2 1), y (2 0 1), x (1 0 1), z (1 3 0), g (2 0 3). Hence the crystal is a combination of the forms $\{0\ 1\ 0\}$, $\{0\ 0\ 1\}$, $\{1\ 1\ 0\}$, $\{1\ 3\ 0\}$, $\{0\ 2\ 1\}$, $\{2\ 0\ 1\}$, $\{1\ 0\ 1\}$, $\{2\ 0\ 3\}$, $\{1\ 1\ 1\}$. It is cleavable parallel to the faces of the forms $\{0\ 0\ 1\}$, $\{1\ 1\ 0\}$, $\{0\ 1\ 0\}$.

Having given tt', pt, px, t, p, x, &c. (fig. 96), being the poles of the faces t, p, x, &c., to determine the positions of the poles q, y, n, o, z.

Let tt', px meet in a. Then a will be (1 0 0). $mt = \frac{1}{2}tt'$, $\cos pt = \sin mt \cos pa$. The right-angled triangles xpo, apt, having a common angle at p, give $\sin pa \cot om = \sin px \cot tm$. $2 \cot pq = 3 \cot px - \cot pa$, $2 \cot py = \cot px + \cot pa$ (27). Tan $mt = 3 \tan mz$ (207). The right-angled triangles tya, $n'yp'$, having a common angle at y, give $\sin ay \cot mn' = \sin p'y \cot mt$. If $tt' = 118^0.49'$, $pt = 67^0.44'$, $px = 60^0.20'$, we shall have $pa = 63^0.53'$, $om = 63^0.7'$, $pq = 34^0.13'$, $py = 80^0.23'$, $mz = 29^0.25'$, $mn' = 45^0.3'$.

220. In a crystal of Oxalic acid (fig. 97), the zones $pacp'$, $pee'p'$ have their axes at right angles to each other. $aem'a'$, $cemc'$ are zones. p, a, c, &c. being the poles of the faces p, a, c, &c. $pa = 50^0.40'$, $pc = 76^0.45$, $mm' = 63^0.5'$. If the zone-circles pa, mm' meet in d, it appears from approximate measurements that $2 \cot pd = \cot pa - \cot pc$ (27). Whence $pd = 73^0.43'$. The right-angled triangles ecp, mcd, having a common angle at c, give $\sin pc \cot pe = \sin dc \cot md$. Whence $pe = 72^0.44'$, $ee' = 34^0.32'$. It frequently happens that there are no faces parallel to the faces e, e'. In this case (204) the form to which the faces e belong is hemihedral.

CHAPTER VII.

DOUBLY OBLIQUE PRISMATIC SYSTEM.

221. In the doubly oblique prismatic system the form $\{h\,k\,l\}$ has the two faces $(h\,k\,l)$ $(\bar{h}\,\bar{k}\,\bar{l})$.

222. To determine the position of any pole.

Let the axes of the crystal meet the surface of the sphere of projection in X, Y, Z (fig. 98). Let A, B, C be the poles of (1 0 0), (1 0 1), (0 0 1), P the pole of $(h\,k\,l)$. XYZ, ABC are supplemental triangles.

$$\cos PX = \sin PBC \sin PB = \sin PCB \sin PC,$$

$$\cos PY = \sin PCA \sin PC = \sin PAC \sin PA,$$

$$\cos PZ = \sin PAB \sin PA = \sin PBA \sin PB.$$

But

$$\frac{a}{h} \cos PX = \frac{b}{k} \cos PY = \frac{c}{l} \cos PZ.$$

Whence

$$\frac{b}{k} \sin PAC = \frac{c}{l} \sin PAB,$$

$$\frac{c}{l} \sin PBA = \frac{a}{h} \sin PBC,$$

$$\frac{a}{h} \sin PCB = \frac{b}{k} \sin PCA.$$

$$\therefore\ \tan \tfrac{1}{2}(PAB - PAC) = \tan \tfrac{1}{2} BAC \tan (45 - \theta),$$

where $\tan\theta = \dfrac{k}{l}\dfrac{c}{b}$;

$$\tan\tfrac{1}{2}(PBC - PBA) = \tan\tfrac{1}{2}CBA\tan(45 - \phi),$$

where $\tan\phi = \frac{l}{h}\frac{a}{c}$;

$$\tan\tfrac{1}{2}(PCA - PCB) = \tan\tfrac{1}{2}ACB\tan(45 - \psi),$$

where $\tan\psi = \frac{h}{k}\frac{b}{a}$.

Whence, knowing the angles A, B, C, the segments into which they are divided by PA, PB, PC become known.

223. Let PX, PY, PZ meet the sides of ABC opposite to A, B, C, in x, y, z. The angles at x, y, z are right angles, therefore

$$\frac{\sin Bx}{\cot PBC} = \frac{\sin Cx}{\cot PCB}, \frac{\sin Cy}{\cot PCA} = \frac{\sin Ay}{\cot PAC}, \frac{\sin Az}{\cot PAB} = \frac{\sin Bz}{\cot PBA}.$$

$$\therefore \tan\tfrac{1}{2}(Bx - Cx) = \frac{\sin(PCB - PBC)}{\sin(PCB + PBC)}\cot\tfrac{1}{2}BC,$$

$$\tan\tfrac{1}{2}(Cy - Ay) = \frac{\sin(PAC - PCA)}{\sin(PAC + PCA)}\cot\tfrac{1}{2}CA,$$

$$\tan\tfrac{1}{2}(Az - Bz) = \frac{\sin(PBA - PAB)}{\sin(PBA + PAB)}\cot\tfrac{1}{2}AB.$$

Whence the segments, into which the sides of ABC are divided by perpendiculars Px, Py, Pz, become known.

$$\cot PA = \cos PAB\cot Az = \cos PAC\cot Ay,$$

$$\cot PB = \cos PBC\cot Bx = \cos PBA\cot Bz,$$

$$\cot PC = \cos PCA\cot Cy = \cos PCB\cot Cx.$$

224. Let PA, PB, PC meet the sides of ABC in H, K, L. The symbols of the points thus determined will be $(0\,k\,l)$, $(h\,0\,l)$, $(h\,k\,0)$.

$$\frac{\cos HY}{\cos HZ} = \frac{\sin HC \sin C}{\sin HB \sin B} = \frac{\sin HC \sin AB}{\sin HB \sin CA},$$

whence

$$\frac{k}{b}\frac{\sin HB}{\sin AB} = \frac{l}{c}\frac{\sin HC}{\sin CA}.$$

Similarly

$$\frac{l}{c}\frac{\sin KC}{\sin BC} = \frac{h}{a}\frac{\sin KA}{\sin AB};$$

and

$$\frac{h}{a}\frac{\sin LA}{\sin CA} = \frac{k}{b}\frac{\sin LB}{\sin BC}$$

Therefore

$$\tan \tfrac{1}{2}(HB - HC) = \tan \tfrac{1}{2}BC \tan (45 - \alpha),$$

$$\text{where } \tan \alpha = \frac{k}{l}\frac{c}{b}\frac{\sin CA}{\sin AB},$$

$$\tan \tfrac{1}{2}(KC - KA) = \tan \tfrac{1}{2} CA \tan (45 - \beta),$$

$$\text{where } \tan \beta = \frac{l}{h}\frac{a}{c}\frac{\sin AB}{\sin BC},$$

$$\tan \tfrac{1}{2}(LA - LB) = \tan \tfrac{1}{2}AB \tan (45 - \gamma),$$

$$\text{where } \tan \gamma = \frac{h}{k}\frac{b}{a}\frac{\sin BC}{\sin CA}.$$

Having determined the segments into which H, K, L divide the sides of ABC, by means of the preceding equations, we can find AH, BK, CL.

$$\sin AP \sin APC = \sin AC \sin PCA,$$

$$\sin HP \sin HPC = \sin HC \sin PCB,$$

$$\sin AP \sin APB = \sin AB \sin PBA,$$

$$\sin HP \sin HPB = \sin HB \sin PBC.$$

$$\frac{a}{h}\sin PCB = \frac{b}{k}\sin PCA, \qquad \frac{c}{l}\sin PBA = \frac{a}{h}\sin PBC.$$

Whence

$$\frac{\sin PH}{\sin PA} = \frac{h}{k}\frac{b}{a}\frac{\sin HC}{\sin CA} = \frac{h}{l}\frac{c}{a}\frac{\sin HB}{\sin AB}.$$

Similarly

$$\frac{\sin PK}{\sin PB} = \frac{k}{l}\frac{c}{b}\frac{\sin KA}{\sin AB} = \frac{k}{h}\frac{a}{b}\frac{\sin KC}{\sin BC};$$

and

$$\frac{\sin PL}{\sin PC} = \frac{l}{h}\frac{a}{c}\frac{\sin LB}{\sin BC} = \frac{l}{k}\frac{b}{c}\frac{\sin LA}{\sin CA}.$$

Therefore

$$\tan \tfrac{1}{2}(PA - PH) = \tan \tfrac{1}{2}HA \tan (45 - \varpi),$$

$$\text{where } \tan \varpi = \frac{h}{k}\frac{b}{a}\frac{\sin HC}{\sin CA} = \frac{h}{l}\frac{c}{a}\frac{\sin HB}{\sin AB};$$

$$\tan \tfrac{1}{2}(PB - PK) = \tan \tfrac{1}{2} KB \tan (45 - \rho),$$

$$\text{where } \tan \rho = \frac{k}{l}\frac{c}{b}\frac{\sin KA}{\sin AB} = \frac{k}{l}\frac{a}{b}\frac{\sin KC}{\sin BC};$$

$$\tan \tfrac{1}{2}(PC - PL) = \tan \tfrac{1}{2}LC \tan (45 - \sigma),$$

$$\text{where } \tan \sigma = \frac{l}{h}\frac{a}{c}\frac{\sin LB}{\sin BC} = \frac{l}{k}\frac{b}{c}\frac{\sin LA}{\sin CA}.$$

Whence the segments into which AH, BK, CL are divided by P become known.

In the preceding investigations we have supposed the indices of P, and of the nearest poles of the forms $\{1\,0\,0\}$, $\{0\,1\,0\}$, $\{0\,0\,1\}$ to be all positive. Therefore, when any of the indices of P are negative, we must change their signs and the signs of the corresponding indices in the symbols of the other poles of the crystal.

225. To find the distance between any two poles.

Let P, Q be the two poles. Then, having found the distances of P, Q from one of the angles of the triangle ABC, and the angles these distances make with one of the adjacent

sides of ABC, and, therefore, the angle they make with each other, we have two sides of a spherical triangle and the included angle, from which the third side PQ may be found.

226. To determine the inclinations of the axes and the ratios of the parameters.

When the distances of A, B, C, the poles of (1 0 0), (0 1 0), (0 0 1), from each other, and the distance of any two of them from P, the pole of $(h\,k\,l)$, are known, the distance of the third from P, as well as the angles which PA, PB, PC make with the sides of the triangle ABC may be found. Cos PX, $\cos PY$, $\cos PZ$ may then be found in terms of PA, PB, PC, and of the angles into which A, B, C are divided by PA, PB, PC. The ratios of the parameters are then given by the equations

$$\frac{a}{h}\cos PX = \frac{b}{k}\cos PY = \frac{c}{l}\cos PZ.$$

The triangles ABC, XYZ are supplemental to each other;

$$\therefore\ YZ = 180^0 - A,\ ZX = 180^0 - B,\ XY = 180^0 - C.$$

227. Having given the symbols of four poles D, E, F, G (fig. 99), and five of the arcs DF, FE, EG, GD, DE, FG; to find the inclinations of the axes and the ratios of the parameters.

Let A, B, C be the poles of (1 0 0), (0 1 0), (0 0 1). Let DE, FG intersect in H, and let them meet the sides of ABC respectively opposite to A, B, C in L, M, P, Q, R, S. Five of the arcs DF, FE, EG, GD, DE, FG being known, we are enabled to calculate the sixth, and also the arcs DH, HE, FH, HG, and the angles they make with each other. The symbols of D, E, F, G, A, B, C being known, the symbols of H, L, M, P, Q, R, S may be found by (17). Hence DH, HE, GH, HF being known, we can find HL, HM, HP, HQ, HR, HS, and thence the sides of the triangle ABC, and the distances of H from its angles. This being done, YZ, ZX, XY, and the ratios of a, b, c may be found by (226).

Examples.

228. In a crystal of Axinite (fig. 100), the following zones have been observed; $mpdfem'$, $mltvwm'$, $mrsxycm'$, $mgom'$, $focwnf'$, $fgyvf'$, $fxtf'$, $fslf'$, $ecqve'$, $pyqwp'$, $pxvnp'$, $pstp'$, $prlp'$.

Let m be $(1\,0\,0)$, f $(0\,1\,0)$, v $(0\,0\,1)$, x $(1\,1\,1)$. Then (17), y will be $(0\,1\,1)$, t $(1\,0\,1)$, p $(1\,1\,0)$, e $(\bar{1}\,1\,0)$, s $(2\,1\,1)$, l $(2\,0\,1)$, r $(3\,1\,1)$, q $(\bar{1}\,1\,2)$, c $(\bar{1}\,1\,1)$, w $(\bar{1}\,0\,1)$, n $(\bar{1}\,\bar{1}\,1)$, o $(\bar{1}\,2\,1)$, g $(0\,2\,1)$. If the distances between m, p, d, f, the poles of the faces m, p, d, f, be measured, it will be found that $\cot md - \cot mf = 2\,(\cot mp - \cot mf)$. Therefore (27), if $(u\,v\,w)$ be the symbol of d, $u = 2$, $v = 1$. d is in the zone-circle mf, therefore $w = 0$; therefore d is $(1\,2\,0)$.

Let m, t, p, &c. (fig. 101) be the poles of the faces m, t, p, &c. Having given $mx = 49^0.\,32'$, $xy = 29^0.\,52'$, $mt = 44^0.\,35'$, $tv = 32^0.\,55'$, $yv = 40^0.\,51'$; to determine the places of the remaining poles.

In the triangle ymv, the sides of which are known, we find $ymv = 44^0.\,41',5$, $yvm = 88^0.\,10',5$. xm, tm, xmt give $xtm = 84^0.\,14',5$. tv, fvt, ftv give $fv = 97^0.\,36'$. fv, vm, fvm give $vfm = 97^0.\,58',5$, $fm = 89^0.\,55'$. vm, mx, vmx give $xvm = 44^0.\,44'$. vm, xvm, vmp give $mp = 45^0.\,12'$. The formula expressing the relation between the distances of four poles in the same zone-circle (27) gives

$\cot fe' = 2\cot fm - \cot fp$; $\cot mw = 2\cot mv - \cot mt$; $\cot mc = 2\cot my - \cot mx$. Whence $fe' = 135^0.\,12'$, $mw = 119^0.\,50'$, $mc = 115^0.\,35'$. mw, mp, wmp give $pw = 115^0.\,50',5$. my, mp, ymp give $py = 58^0.\,53'$. $2\cot pq = \cot pw + \cot py$;

$$2\cot md = \cot mf + \cot mp;\quad \cot me = 2\cot mf - \cot mp;$$

$$\cot fg = 2\cot fy - \cot fv;\quad \cot ms = 2\cot mx - \cot my;$$

$$\cot mr = 3\cot mx - 2\cot my;\quad \cot ml = 2\cot mt - \cot mv;$$

$4\cot mk = 5\cot mt - \cot mv$. Whence $pq = 86^0.\,35'$, $md = 63^0.\,34'$, $me = 134^0.\,43'$, $fg = 34^0.\,53',5$, $ms = 33^0.\,20',5$, $mr = 25^0.\,27'$, $ml = 28^0.\,57'$, $mk = 39^0.\,30'$.

If the axes meet the surface of the sphere of projection in X, Y, Z, mfv will be the polar triangle of XYZ, and YZ, ZX, XY will be the supplements of the angles fmv, vfm, fvm (226); fv, vm, mf give $vfm = 78^0.16'$. Hence

$$YZ = 82^0.1',5, \quad ZX = 101^0.44', \quad XY = 91^0.49',5.$$

Let a, b, c be the parameters of the crystal. Then, since x is (1 1 1), $a \cos xX = b \cos xY = c \cos xZ$. $\cos xX = \sin xv \sin xvf$, $\cos xY = \sin xv \sin xvm$, $\cos xZ = \sin xm \sin xmf$. Whence

$$\frac{2,023}{a} = \frac{1,976}{b} = \frac{1,580}{c}.$$

To find the symbols of the faces when referred to the zones mp, pt, tm as crystallographic axes.

The symbols of the three zones when referred to the old axes are [0 0 1], [$\bar{1}$ 1 1], [0 1 0]. Therefore (28), if u, v, w be the indices of any face when referred to the old axes, u', v', w' its indices when referred to the new axes, $u' = w$, $v' = -u+v+w$, $w' = v$. Hence the symbols of the faces will be f (0 1 1), m (0 1 0), t (1 0 0), v (1 1 0), w (1 2 0), l (1 $\bar{1}$ 0), p (0 0 1), e (0 2 1), o (1 4 2), y (1 2 1), c (1 3 1), x (1 1 1), s (1 0 1), r (1 $\bar{1}$ 1), n (1 1 $\bar{1}$), q (2 4 1), g (1 3 2), d (0 1 2).

229. In a crystal of Blue Vitriol (fig. 103) the zones are $mntrm'$, $rvkoqwr'$, $rxpr'$, $mpvm'$, $npkn'$, $tpot'$, $txvt'$. Let k be (0 0 1), n (0 1 0), v (1 0 1), m ($\bar{1}$ 1 0). Then (17) r will be (1 0 0), and p will be (0 1 1). If the distances between the poles of r, v, k, o, q, w be measured, it will be found that, r, v, k, &c. being the poles of the faces r, v, k, &c, $\cot rv - \cot rk = -(\cot ro - \cot rk) = -\frac{1}{2}(\cot rq - \cot rk) = -\frac{1}{3}(\cot rw - \cot rk)$. Therefore (27) o is ($\bar{1}$ 0 1), q is ($\bar{2}$ 0 1), and w is ($\bar{3}$ 0 1). Hence (17) t is (1 1 0), x (2 1 1). h, s, z (fig. 104) are the poles of faces which do not occur in the crystal (fig. 103). Their symbols are h (2 1 0), s (1 1 1), z (3 1 1).

Having given nt, tr, nk, pt, pr to find the positions of the remaining poles.

rt, rn give rh (26) or (27). pt, tr, rp give ptr and prt. pt, tn, ptr give pn and pnt. kn, rn, pnt give kr and krn. krn, ptr, tr give ro. kr, rh, krn give khr. khr, prt, rh give rx. rx, rp give rs, rz (26) or (27). In like manner, rk, ro give rv, rq, rw. The values of the five distances which we have supposed to be used in calculating the places of the poles are $nt = 30^0.51'$, $tr = 69^0.50'$, $nk = 109^0.38'$, $pt = 52^0.20'$, $pr = 76^0.33'$.

The three zone-circles rt, rp, rk pass through all the poles of the crystal. The angles which the zone-circles make with each other, and the distance of each pole from r are known. Therefore the distance between any two poles, being one side of a spherical triangle of which two sides and the included angle are known, may be easily calculated.

If the axes meet the surface of the sphere of projection in X, Y, Z,

$$YZ = 180^0 - nrk, \quad ZX = 180^0 - knr, \quad X = 180^0 - nkr.$$

Let a, b, c be the parameters. Then since t, p are (1 1 0), (0 1 1) respectively, $a \cos tX = b \cos tY$, $b \cos pY = c \cos pZ$. $\cos tX = \sin tn \sin tnp$, $\cos tY = \sin tr \sin trk$, $\cos pY = \sin pr \sin prk$, $\cos pZ = \sin pr \sin prt$. Whence

$$\frac{1}{a}\frac{\sin tr \sin trk}{\sin tn \sin tnp} = \frac{1}{b} = \frac{1}{c}\frac{\sin prk}{\sin prt}.$$

CHAPTER VIII.

TWIN CRYSTALS.

230. A TWIN crystal is composed of two crystals joined together in such a manner that one would come into the position of the other by revolving through two right angles round an axis which is perpendicular to a plane which either is, or may be, a face of either crystal. The axis will be called the twin axis, and the plane to which it is perpendicular the twin plane.

231. Let the poles of the two crystals be projected upon the same sphere (fig. 104). Let T, T' be the extremities of a diameter perpendicular to the twin plane. Let P, p be the poles of any corresponding faces of the two crystals, p' the pole opposite to p. Since p may be made to coincide with P by turning the crystal through 180^0 round TT', PTp must be an arc of a great circle bisected in T. In like manner if Q, q be the poles of any other corresponding faces of the two crystals, Qq will be bisected in T. Hence the arcs joining the poles of any corresponding faces of two crystals forming a twin crystal, are bisected by the poles of the twin face. If p', q' be the poles of the faces opposite to p, q respectively, it is manifest that Pp', Qq' are bisected at right angles by the great circle MN having T, T' for its poles. Hence the poles of the opposite faces of the two crystals are symmetrically arranged with respect to a great circle the plane of which is parallel to the twin face.

232. In order to find the twin axis in any given twin crystal, when it cannot be done by simple inspection, we must determine by measurement or by the observation of zones, the

intersection of two great circles each of which passes through the poles of corresponding or opposite faces of the two crystals. If the intersections of the circles be the poles of corresponding faces of the two crystals, they will be the poles of a twin plane. Let P, Q (fig. 104) be the poles of any two faces of one crystal; p, q the poles of the corresponding faces of the other, p', q' the poles of the faces opposite to p, q; T, T' the intersections of the great circles pPp', qQq'. Then, if T be the pole of corresponding faces of the two crystals, the triangles PTQ, pTq are equal and similar, and therefore p, q may be made to coincide with P, Q by turning the crystal to which p, q belong through 180^0 round TT'. Hence T, T' are the poles of the twin plane.

233. When the twin plane, and the angles between the faces of one of the crystals are given, the angle between any faces of each of the two crystals can be readily determined.

Let P, P' be the poles of opposite faces of the two crystals. Then $PT = PT'$, therefore $PP' = 180^0 - 2PT$. When PT is greater than 90^0, the faces P, P' will form a re-entering angle. Let P, Q' be any faces of each of the two crystals, PT, QT, PQ being known, PTQ may be found. $TQ' = 180^0 - TQ$. PT, $Q'T$, PTQ being known, PQ' may be found.

Examples of Twin Crystals belonging to the Octahedral System.

234. In twin crystals belonging to the octahedral system, the twin axis is either perpendicular to a face of $\{1\,1\,1\}$ or to a face of $\{0\,1\,1\}$.

235. In a twin crystal of Magnetic Iron Oxide (fig. 105), the two octahedrons are joined in such a manner that the face o of one crystal is parallel to the face $o_{,}$ of the other, and the faces o, o', $o'_{,}$, $o_{,}$ belong to the same zone. One crystal will evidently come into the position of the other by revolving through 180^0 round an axis perpendicular to the faces o, $o_{,}$, and which is, therefore, the twin axis. o, o', $o'_{,}$, $o_{,}$ being the

poles of the faces denoted by the same letters, $oo' = 70^0.31',7$, $o'o_{\prime}' = 180^0 - 2oo' = 38^0.56',5$, $o''o = 109^0.28',3$, $o''o_{\prime}'' = 180^0 - 2oo'' = -38^0.56',5$. The faces o'', $o_{\prime}''$ form a re-entering angle.

236. In a twin crystal of Zinc Blende (fig. 106), the individuals are dodecahedrons. Six faces of one crystal coincide with six faces of the other. One crystal will come into the position of the other by revolving through 180^0 round a line parallel to the intersection of d, d', and which is perpendicular to a face of the form $\{1\,1\,1\}$. Hence this line is the twin axis.

237. In a twin crystal of Fluor (fig. 107) the individuals are cubes. If a, a', a'', $a_{\prime}$, $a_{\prime}'$, $a_{\prime}''$ be the poles of the faces denoted by the same letters, it is found that $aa_{\prime}' = a_{\prime}a'$, $a''a_{\prime}'' = 109^0.28',3$. The arcs $aa_{\prime}$, $a'a_{\prime}'$ joining the poles of corresponding faces of the two crystals intersect in o, the bisection of the arc $a''a_{\prime}''$. $oa_{\prime}'' = 54^0.44',8$, and oa'' evidently bisects the angle $aa''a'$. Therefore o is the pole of a face of the octahedron. Therefore the twin axis is a normal to that face of $\{1\,1\,1\}$ which truncates the solid angle formed by the faces a, a', a''.

238. In a twin crystal of Gold (fig. 108), the individuals are the icositessarahedrons $\{3\,1\,1\}$. The zones pr, sq of one crystal coincide with the zones $p'r'$, $s'q'$ of the other. But the face of the octahedron adjacent to the faces m, r, s is common to the zones pr, qs. Therefore the great circle through the poles of p, p' intersects the great circle through the poles of q, q', in o, the pole of that octahedron, a normal to which is, therefore, the twin axis. Then, p, q, p', q' being the poles of the faces p, q, p', q', we have $pp' = 180^0 - 2po = 20^0.4'$, $qq' = -20^0.4'$. The faces qq' form a re-entering angle.

239. A twin crystal of Diamond (fig. 109) is composed of two hemioctahedrons, the faces of which are parallel to the alternate faces of the same octahedron. One half form comes into the position of the other by revolving through 180^0 round a normal to a face of the form $\{0\,1\,1\}$, which is therefore the twin axis.

240. A twin crystal of Yellow Iron Pyrites (fig. 110) is composed of two hemitetrakishexahedrons, the faces of which are parallel to the faces of the same holohedral form. It is easily seen by inspecting fig. 10, that the poles of one form will coincide with those of the other after revolving through 180^0 round any two opposite poles of $\{0\,1\,1\}$. Therefore the twin axis is a normal to any face of $\{0\,1\,1\}$.

241. In the first four of the preceding examples we have found the twin axis to be perpendicular to a face of the octahedron. This is perhaps the only way in which the individuals of this system composing twin crystals are united. Twins analagous to the last two examples, in which the twin axis is perpendicular to a face of $\{0\,1\,1\}$, can only be produced by the union of hemihedral forms. In such crystals, (Mohs Naturgeschichte des Mineralreichs 154—158), the crystallographic axes of one of the presumed individuals are parallel to those of the other, and the cleavages of one are parallel to those of the other, or continued into it without interruption. We cannot therefore determine with certainty whether such crystals are to be considered as twins or as single crystals the faces of which are repeated with a certain degree of regularity.

Examples of Twin Crystals belonging to the Pyramidal System.

242. In twin crystals belonging to the pyramidal system, the twin axis is perpendicular to a face of one of the forms $\{1\,0\,0\}$, $\{1\,1\,0\}$, $\{h\,0\,l\}$, $\{h\,h\,l\}$.

243. In a twin crystal of Oxide of Tin (fig. 111), the two pyramidal faces s, s' of one crystal are respectively parallel to the corresponding faces $s_{\prime}'$, $s_{\prime}$ of the other. One crystal comes into the position of the other by making a half revolution round an axis perpendicular to the face which belongs to the zone $s\,s'$, and makes equal angles with the faces s, s'. This line, therefore, is the twin axis. Let c be the pole of $(0\,0\,1)$, t the pole of the face which truncates the edge formed by s, s'. $cs = 43^0.\,38'$; whence (112) $ct =$

$33^0.\,59'$, $ts = 29^0.\,12'$, $ts'' = 71^0.\,23'$, $tg = 63^0.\,35'$, $tg'' = 119^0.\,12'$. Whence $gg_{,} = 52^0.\,50'$, $g''g_{,}'' = -\,58^0.\,12'$, $ss_{,} = 121^0.\,36'$, $s''s_{,}'' = 37^0.\,14'$.

If we assume the symbols of the faces s, s' to be (1 1 1), ($1\,\bar{1}\,1$), the symbol of t will be (1 0 1). If we assume s, s' to be (1 0 1), (0 1 1), t will be (1 1 2).

244. In a twin crystal of Copper Pyrites (fig. 112), the zone pp' coincides with the zone $p_{,}p_{,}'$, and the face p is parallel to the face $p_{,}$. In this case the twin axis is manifestly perpendicular to p, a face of the form $\{1\,1\,1\}$.

If p, p', p'', &c. be the poles of the faces p, p', p'', &c., $pa = 54^0.\,20'$, $pp'' = 108^0.\,40'$, $pp' = 70^0.\,7'$, $pp''' = 109^0.\,53'$, $pb = 29^0.\,45'$, $pc = 39^0.\,5',5$. Hence $aa_{,} = 71^0.\,20'$, $p''p_{,}'' = -\,37^0.\,20'$, $p'p_{,}' = 39^0.\,46'$, $p'''p_{,}''' = -\,39^0.\,46'$, $bb_{,} = 20^0.\,30'$, $cc_{,} = 109^0.\,11'$.

245. The individuals composing a twin crystal of Copper Pyrites (fig. 113) are square pyramids, having a hemihedral character in consequence of the very unequal size of their alternate faces. The large faces of one crystal are parallel to the small faces of the other. One crystal will evidently come into the place of the other by revolving through 180^0, round an axis perpendicular to any one of the faces of a square prism truncating the edges pp, $p'p'$. The twin axis will, therefore, be a normal to a face of $\{1\,0\,0\}$ or $\{1\,1\,0\}$, according as we consider p to be a face of $\{1\,0\,1\}$ or of $\{1\,1\,1\}$.

246. In a twin crystal of Scheelale of Lime (fig. 114), the individuals are similar to the crystal represented in fig. 55, and are joined so that the faces p of the two crystals coincide with each other. One crystal will come into the place of the other by revolving through 180^0 round an axis perpendicular to any one of the faces of a square prism truncating the edges pp, $p'p'$, or nn. Hence the twin axis is a normal to a face of either of the forms $\{1\,0\,0\}$, $\{1\,1\,0\}$.

247. In the last two examples the axes and cleavage planes of the two crystals are parallel. The propriety of con-

sidering them twin crystals is, therefore, doubtful. On the other hand, in the last example, the hemihedral forms are dissimilar, and striæ, which run parallel to the intersection of a with p, meet in a line which divides the faces $pp_{,}$, $p'p_{,}'$ into two parts each of which appears to belong to a different crystal.

Examples of Twin Crystals belonging to the Rhombohedral System.

248. In twin crystals belonging to the rhombohedral system, the twin axis is either perpendicular to a face of $\{1\,1\,1\}$, or to a face of one of the rhombohedrons.

249. In a twin crystal of Calc Spar (fig. 115), the individuals of which are combinations of the forms $\{0\,1\,1\}$ (g), $\{1\,1\,\bar{2}\}$ (c), the faces c of the two crystals coincide with each other. One crystal comes into the position of the other by revolving through two right angles round a line parallel to the intersection of c, c'. But this line is perpendicular to the face $(1\,1\,1)$, and is, therefore, the twin axis.

250. The individuals composing the twin crystal of Calc Spar (fig. 116), are rhombohedrons, the faces of which are parallel to the cleavage planes. The two obtuse angles which the faces of one crystal make with those of the other are equal; and the angle between normals to the two faces p, $p_{,}$ is equal to twice the angle between a normal to p, and a normal to the face $(1\,1\,1)$. Hence one crystal will come into the place of the other by revolving through two right angles round a normal to $(1\,1\,1)$, which, therefore, is the twin axis.

251. The individuals composing the twin crystal of Calc Spar (fig. 117), are combinations of the forms $\{1\,1\,1\}$ (o), $\{1\,1\,\bar{2}\}$ (c). The zone co coincides with the zone $c_{,}o_{,}$. $cc = 52^0.\,30',5$. Let the arcs $cc_{,}$, $c'c_{,}'$, through the poles of opposite faces of the two crystals meet in t, t'. $t'c_{,} = 90 - \frac{1}{2}cc_{,} = 63^0.\,45'$. $ot = 90^0 - t'c_{,} = 26^0.\,15'$. Therefore t is the pole of a face of the form $\{0\,1\,1\}$. Hence the twin axis is perpendicular to a face of the form $\{0\,1\,1\}$.

252. In the twin crystal of Calc Spar (fig. 118), the individuals of which are combinations of $\{1\,1\,\bar{2}\}$ (c), $\{2\,0\,1\}$ (r), $\{1\,1\,0\}$ (g), $\{1\,1\,\bar{1}\}$ (f), the zone gc in one crystal coincides with the zone $g_{,}c_{,}$ in the other, and the distance between the poles of g, $g_{,} = 38^0.\,16',4$. Let t be the intersection of the circles $gg_{,}$, $rr_{,}$ drawn through the poles of opposite faces of the two crystals. Then $tg = 90^0 - \frac{1}{2}gg_{,} = 70^0.\,51',8$, therefore t is the pole of a face of the cleavage rhombohedron $\{1\,0\,0\}$. Hence the twin axis is perpendicular to a cleavage plane.

c, g, r, f being the poles of the faces c, g, r, f, $tc = 45^0.\,23',4$, therefore $cc_{,} = 89^0.\,13',2$. $tc' = 68^0.\,57'$, therefore $c'c_{,}' = 42^0.\,6'$. $tg' = 37^0.\,27',5$, therefore $g'g_{,}' = 105^0.\,5'$. $tf = 107^0.\,44'$, therefore $ff_{,} = -\,35^0.\,28'$. $tf' = 50^0.\,34',5$, therefore $f'f_{,}' = 78^0.\,51'$.

253. The individuals composing a twin crystal of Calc Spar (fig. 119) have the form $\{2\,0\,\bar{1}\}$. The faces r, r' of one crystal are respectively parallel to the faces $r_{,}$, $r_{,}'$ of the other. The places of r, r' are interchanged, and one crystal comes into the position of the other by revolving through two right angles round a normal to a face of $\{1\,1\,\bar{1}\}$, the pole of which bisects the arc joining the poles of r, r'. Hence the twin axis is perpendicular to that face of $\{1\,1\,\bar{1}\}$ which truncates the edge formed by r, r'.

254. In a twin crystal of Ruby Silver (fig. 120), the faces z, z' of one crystal coincide with the faces $z_{,}$, $z_{,}'$ of the other. One crystal will therefore come into the place of the other by revolving through two right angles round a perpendicular to a face, the pole of which bisects the arc joining the poles of z, z'. The poles of z bisect the arcs joining the adjacent poles of the cleavage planes. $zz' = 42^0.\,21'$. If we make the cleavage rhombohedron $\{1\,0\,0\}$, z, z' will be poles of the form $\{0\,1\,1\}$, and the twin axis will be perpendicular to a face of $\{2\,1\,1\}$.

Examples of Twin Crystals belonging to the Prismatic System.

255. In a twin crystal of Aragonite (fig. 121), the zone mm' of one crystal coincides with the zone $m_{,}m_{,}'$ of the other,

and the faces m, $m_{,}$ of the two crystals are parallel. Hence one crystal will come into the place of the other by revolving through two right angles round a normal to m, which is, therefore, the twin axis.

m, h, k being the poles of the faces m, h, k, we have $mm' = 63^0.\ 50'$, $mh = 58^0.\ 5'$, $hk = 54^0.\ 13'$. $\cos mk = \cos mh \cos hk$, therefore $mk = 71^0.\ 59,5$, $mk' = 108^0.\ 0',5$. Whence $m'm_{,}' = 52^0.\ 20'$, $hh_{,} = 62^0.\ 50'$, $kk_{,} = 36^0.\ 1'$, $k'k_{,}' = -36^0.\ 1'$.

256. In a twin crystal of Staurolite (fig. 122), the zone or of one crystal coincides with the zone $o_{,}r_{,}$ of the other, and the distance between the poles of $oo_{,} = -60^0.\ 36'$. $mm' = 50^0.\ 40'$, $pr = 55^0.\ 22'$. From these data it appears that, if we make o (1 0 0), p (0 0 1), m (1 1 0), r (0 1 1), a point in the great circle $oo_{,}$, 90^0 from the bisection of $oo_{,}$, and therefore $120^0.\ 18'$ from o, in the zone-circle or, will be the pole of (3 2 2). Hence the twin axis is perpendicular to the face (3 2 2).

Let t be the pole of the twin face, m, p, r the poles of the faces m, p, r, $tm = 64^0.\ 46'$, $tm' = 115^0.\ 14'$, $tp = 60^0.\ 37'$, $tr = 30^0.\ 18'$. Whence $mm_{,} = 50^0.\ 28'$, $m'm_{,}' = -50^0.\ 28'$, $pp_{,} = 58^0.\ 46'$, $rr_{,} = 119^0.\ 24'$.

257. In a twin crystal of Staurolite (fig. 123), the zone po of one crystal coincides with the zone $p_{,}o_{,}$ of the other, and the distance between the poles of o, $o_{,} = -88^0.\ 24'$. Hence a point bisecting the arc joining the poles of o, $o_{,}$, and therefore $45^0.\ 48'$ from o, in the zone-circle po, is the pole of (3 0 2). Hence the twin axis is perpendicular to the face (3 0 2).

Examples of Twin Crystals belonging to the Oblique Prismatic System.

258. In a twin crystal of Felspar (fig. 124), the zones mt, py of one crystal coincide with the zones $m_{,}t_{,}$, $p_{,}y_{,}$ of the other. Hence one crystal will come into the place of the other by revolving through two right angles round an axis perpendicular to a face common to the zones mt, py, which, therefore, is the twin axis.

Let a be the pole of the face to which the twin axis is perpendicular, p, y the poles of the faces p, y. Then $ay = 35^0.44',5$, $ap=116^0.7'$. Therefore $yy_{,}= 108^0.31'$, $pp_{,}=-52^0.14'$.

259. In a twin crystal of Felspar (fig. 125), the zone mt of one crystal coincides with the zone $m_{,}t_{,}$ of the other. The distance between the poles of m, $m_{,}$, opposite faces of the two crystals, is $121^0.10'$. Therefore a point in the great circle $tt_{,}$, 90^0 distant from the bisection of $tt_{,}$, is distant $29^0.25'$ from the pole of m, and therefore is the pole of the face z (fig. 95). Hence the twin axis is perpendicular to the face z.

$zt = 29^0.59'$, $zt' = 91^0.11'$, $zt'' = 88^0.49'$, $zp = 102^0.29'$, $zy = 66^0.31$. Therefore $tt_{,} = 130^0.2'$, $t't_{,}' = -2^0.22'$, $t''t_{,}'' = 2^0.22'$, $pp_{,} = -24^0.58'$, $yy_{,} = 46^0.58'$.

260. In a twin crystal of Felspar (fig. 126), the zones pm, oy of one crystal coincide respectively with the zones $p_{,}m_{,}$, $o_{,}y_{,}$ of the other. Therefore one crystal will come into the position of the other, by revolving through two right angles round an axis perpendicular to the face n (fig. 95), common to the zones pm, oy, which is therefore the twin axis.

p, t, m, n, being the poles of the faces p, t, m, n, we have $pn = 44^0.56',5$, $tn = 84^0.46'$, $mn = 45^0.3',5$. Therefore $pp_{,} = 90^0.7'$, $tt_{,} = 10^0.28'$, $mm_{,} = 89^0.53'$. $pp_{,}$ is found to be very nearly 90^0, and therefore mn' very nearly 45^0. The difference between this and the value of mn given above, is probably due to a small error in the determination of some of the angles from which mn was computed.

261. In a twin crystal of Felspar (fig. 127), the faces m, p of one crystal are respectively parallel to the faces $m_{,}$, $p_{,}$ of the other. One crystal will come into the position of the other by revolving through two right angles round a normal to p, which, therefore, is the twin axis.

p, t, z, x being the poles of the faces p, t, z, x, we have $px = 50^0.19'$, $pt = 112^0.16'$, $pz = 102^0.29'$. Therefore $xx_{,} = 79^0.22'$, $tt_{,} = -44^0.32'$, $zz_{,} = -24^0.58'$.

Example of a Twin Crystal belonging to the Doubly Oblique System.

262. In a twin crystal of Cleavlandite (fig. 128), the zone lmt of one crystal coincides with the zone $l_{,}m_{,}t_{,}$ of the other, and the faces m, $m_{,}$ are parallel. Hence one crystal will come into the position of the other by revolving through two right angles round a normal to m, which, therefore, is the twin axis.

Let p, m, t, l be the poles of the faces p, m, t, l, the faces p, m being parallel to the two most perfect cleavages. It is found that $mt = 62^0.\ 7'$, $ml = 60^0.\ 8'$, $mp = 93^0.\ 36'$. Therefore $tt_{,} = 54^0.\ 56'$, $ll_{,} = 59^0.\ 44'$, $pp_{,} = -\ 7^0.\ 12'$.

263. Occasionally several crystals are joined together in such a manner that every two adjacent crystals are united according to the law stated in (230). In many twin crystals, as the preceding examples shew, faces of the two individuals make with each other re-entering angles. The occurrence, however, of such angles is not always to be taken for a character of a twin crystal; for two or more faces of a simple crystal may be repeated, and thus form a re-entering angle. In some instances the faces are repeated several times, forming a corresponding number of parallel grooves, which, when the faces are very narrow, produce the striæ observed upon the faces of certain forms of some crystals.

CHAPTER IX.

GONIOMETERS, &c.

264. CARANGEAU's Goniometer consists of two small metal rulers (fig. 129), fastened together by a pin so as to move stiffly on each other. In order to measure the mutual inclination of two faces of a crystal, the rulers are placed with their edges touching the faces of the crystal in lines perpendicular to the intersection of the faces. The rulers are then applied to a graduated semi-circle (fig. 130), without altering their position with respect to each other, so that the intersection of their edges may coincide with the center of the graduation. The portion of the graduated arc, contained between the edges of the rulers, will then measure the inclination of the planes, or the supplement of the distance between their poles. This instrument is obviously incapable of affording accurate results.

265. The Reflective Goniometer of Wollaston is represented in fig. 131. A graduated circle L, the divisions of which may be read off to minutes by means of a vernier N, is fixed on a hollow axle which may be turned round by the milled head M. An axle CS passing through the hollow axle of L, and which either turns with the circle L, or may be turned independently by means of the milled head at S, carries a crooked arm CF. The part FG is connected with CF by a joint which permits it to turn round an axis perpendicular to CS, passing through CS produced, and has a collar at G in which the pin HK turns and slides with its axis perpendicular to the axis of FG, and passing through the point in which SC produced intersects the axis of FG. The crystal is fastened by means of a soft cement to a thin plate of metal, fixed in a slit at K.

266. To measure the angle between two faces of a crystal.

Let p, q be two faces of a crystal, (p,q) the angle between lines drawn from any point within the crystal perpendicular to p, q respectively, or the distance between the poles of p, q. Make the intersection of p, q parallel to the axis of the circle, and as nearly coincident with it as possible, by means of the angular motion of FG, and the angular and sliding motion of HK. Place the instrument upon a firm stand; and let A, B (fig. 132) be two signals in a plane passing through a point C in the intersection of the faces p, q, and perpendicular to the axis of the circle. Turn the circle till the image of one of the signals A, seen by reflexion in the face p, coincides with B viewed directly, and read off the arc at which zero of the vernier stands. Now turn the circle till the image of A, seen by reflexion in q, coincides with B seen by direct vision, and read off the arc at which zero of the vernier stands. The difference of the two readings will measure the angle (p,q), and will, therefore, be the supplement of the angle between the faces p, q, according to the usual definition of the angle between two planes that bound a solid.

For, if CE be drawn bisecting ACB, and CD perpendicular to CE in the plane ACB, CD will be perpendicular to p at the first observation, and to q at the second. Hence the crystal, and, consequently, the circle to which it is attached, must have been turned in the interval between the observations through the angle (p,q) contained between perpendiculars to the faces p, q.

267. If a face r belong to the zone of p, q, it will be parallel to the axis of the circle, and, therefore, in some one position of the circle, the image of A seen by reflexion in r will coincide with B seen directly. Hence in order to find the faces which belong to the zone containing two given faces, we must adjust the crystal as for the purpose of measuring the angle between those faces, and then, while the circle makes

one revolution, observe the faces that afford by reflexion images of A passing through B.

268. In order to make ABC, the plane through the two signals and the crystal, perpendicular to the axis of the circle, turn the stand of the instrument, or move A, till the image of A seen by reflexion in the plane surface of the circle, or in any bright plane surface fixed parallel to it, coincides with a point A' seen directly, the distance of which from A, in a line parallel to the axis, is equal to twice the distance of the crystal from the plane of the circle, or from the reflecting surface. Let a solid having two bright parallel surfaces be fixed in the place of the crystal, and adjusted till the image of A seen by reflexion in either surface, describes the same path while the circle revolves, and place the lower signal B in the path traced out by the image of A. Having thus made the plane ABC perpendicular to the axis of the instrument, the intersection of the faces p, q is known to be parallel to the axis, when, on turning the milled head S, the images of A, seen by reflexion in p, q, are observed to pass through B. The adjustment of the edge p, q is most easily made by cementing the crystal to the plate K with one of the two faces, p for example, nearly parallel to the plate, which is to be fixed in the slit at the end of HK, so that intersection of p, q may be nearly perpendicular to HK, and, therefore, HK nearly perpendicular to CS. By turning HK round its axis, the path of the image of A seen in p, may be made to pass through B; and then by turning FG on its axle at F, the path of the image of A seen in q may be made to pass through B. Should the latter adjustment disturb the former, the process must be repeated.

269. The distances of the two signals from the crystal should be nearly equal, and not less than six or eight feet. The upper signal, when the observations are made in the day time, may be a narrow black bar, placed in the upper part of a window, parallel to the axis of the circle; and the lower signal, a white line on a black ground also parallel

to the axis of the circle. In some cases an image of the sun formed by a lens of short focal length, or a small round hole in a screen through which the light of the sky is seen, may be used with advantage for the upper signal. In observing at night, two narrow slits in screens, through one of which is seen the flame of a candle, and through the other a sheet of paper illuminated by a candle placed behind it, answer extremely well for the upper and lower signals respectively.

270. When the points of p, q at which the reflexions take place are not equally distant from the axis of the circle, the angle through which the circle revolves between the two observations will differ slightly from the angle between normals to p, q.

Let A, B (fig. 133) be the two signals; PE, QE the lines in which p, q meet a plane through A, B perpendicular to the axis of the circle at the first and second coincidences of B with the reflected image of A; P, Q the points at which the reflexions take place; (p,q) the angle between normals to the faces; V the angle, not greater than 180^0, through which the circle revolves between the two observations. Then, if $PEQ = E$, we have $(p,q) = V \mp E$, where the upper or lower sign is to be taken according as the circle is turned in the direction PQE or QPE.

Let AP, BQ meet in D, then $2\,PEQ = 2\,PED + 2\,QED = 2\,EPA - 2\,EDA + 2\,EQB - 2\,EDB = APB - ADB + AQB - ADB = PBQ + PAQ$; $\therefore\ PEQ = \frac{1}{2}$ sum of PAQ, PBQ.

When one of the points P, Q is within the angle which AB subtends at the other, we have $PEQ = \frac{1}{2}$ difference between PAQ, PBQ.

271. To eliminate the error arising from the excentricity of the points P, Q at which the reflexions take place.

Let A, B be equidistant from C, the point in which the axis of the circle meets the plane through A, B perpendicular

to it. When the crystal is on the left of the circle, let V be the angle through which the circle has revolved between the two observations; E the error; P, Q the points at which the reflexions take place. Turn the instrument half round in azimuth so that the crystal may be on the right of the circle, and repeat the observations. Let V' be the angle through which the circle revolves; E' the error; P', Q' the points at which the reflexions take place. We shall now have PQ, $P'Q'$ very nearly equal, and equally inclined to EC, but in contrary directions. Hence $PAQ = P'BQ'$, $PBQ = P'AQ'$, and, therefore, E, E' are very nearly equal. Now the directions in which the circle is turned in its first and second positions are opposed to each other, therefore E, E' will have different signs, therefore V, V' are one greater and the other less than (p,q). Hence if $(p,q) = V \mp E$, $(p,q) = V' \pm E'$. Therefore, since E, E' are very nearly equal, $(p,q) = \frac{1}{2}(V + V')$ very nearly.

When either of the faces is large, it should be blackened over, except at the point where it is intended the reflexion should take place. If this precaution were neglected, we should not be at liberty to assume that $PQ = P'Q'$, and that PQ, $P'Q'$ are equally inclined to CE. Any error that may arise from imperfect centering of the circle, will be eliminated, if the observations at a given face, in the two positions of the instrument, be made with zero of the vernier at points distant nearly 180^0 from each other.

When AC, BC are equal, $AC = EC \cos \frac{1}{2} ACB$ nearly; therefore, if PQ make an angle θ with EC, $AC \sin PEQ = PQ \cos \frac{1}{2} ACB \sin \theta$.

272. In many crystals, not belonging to the octahedral system, Mitscherlich discovered that the angles between certain faces vary slightly with the temperature of the crystal; thus the directions of the cleavages of Calc Spar, which, at the ordinary temperature of the atmosphere, make angles of $105^0.5'$ with each other, become more nearly right angled by $8',5$, when the temperature of the crystal is increased 100^0 C. For the same

increase of temperature a line perpendicular to the face (1 1 1) expands 0,00286, and a line parallel to the face (1 1 1) contracts 0,00056 of its length. The distance between the poles of the faces m, m' of a crystal of Aragonite (fig. 84), is diminished $2',8$, and the distance between the poles of the faces k, k' is increased $5',5$ by increasing the temperature of the crystal 100^0 C. In a crystal of Gypsum, which belongs to the oblique prismatic system, the angle between the two axes to which the third axis is perpendicular, and the ratios of the parameters, vary with a change of temperature.

273. To find the plane angles of a face of a crystal.

Let the face p meet the faces q, r; and let P, Q, R be the poles of the faces. The edges in which p meets q, r are perpendicular to the planes of the great circles PQ, PR. Therefore the plane angle formed by the edges in which p intersects q, r is the supplement of the angle QPR. The lengths of the edges are not subject to any known law.

274. The system to which any given crystal belongs, is best determined by observing the kind of symmetry that prevails in the distribution of the poles of its faces. When the crystal is transparent, its optical properties frequently help to determine its system. Hauy ascertained that crystals belonging to the octahedral system have no double refraction. Sir David Brewster discovered that crystals belonging to the pyramidal system have one optic axis parallel to the crystallographic axis OZ, which is perpendicular to the axes along which the two equal parameters are measured: that crystals belonging to the rhombohedral system have one optic axis which makes equal angles with the crystallographic axes; and that crystals belonging to the three remaining systems have two optic axes. In crystals belonging to the prismatic system, the optic axes lie in a plane through two of the crystallographic axes, and the angle they make with each other is bisected by one of the crystallographic axes. In crystals belonging to the oblique prismatic system, the optic axes are either in the plane of the two crystallographic axes OZ, OX

which are perpendicular to the third OY, or in a plane passing through OY, and make equal angles with OY.

Thus in Idocrase the optic axis is perpendicular to the face p (fig. 51). In Calc Spar the optic axis is perpendicular to the face o (fig. 68). In Apatite it is perpendicular to the face p (fig. 73). In Quartz it is parallel to the intersection of the faces r. A crystal of Quartz which has faces of a hemihedral form in the zones containing two adjacent faces p, z, is found to be optically right-handed or left-handed according as the zones are direct, as in fig. 75, or inverse. (Transactions of the Cambridge Philosophical Society, Vol. I. p. 43. Vol. IV. p. 79.) Aragonite has two optic axes perpendicular to the axis of the zone kk' (fig. 84), making angles of $9^0.7'$ with the axis of the zone mm'. In Sulphate of Magnesia the optic axes are perpendicular to the axis of the zone mm' (fig. 86), and make angles of $25^0.50'$ with a normal to the face e. In Saxon Topaz the optic axes make angles of $31^0.9'$ with a normal to p, in a plane through a normal to p and the axis of the zone nn'. In Epidote the optic axes are perpendicular to the axis of the zone mt (fig. 93); one makes an angle of $5^0.11'$ with a normal to r towards a normal to l; the other makes an angle of $18^0.5'$ with a normal to m towards a normal to t. In Felspar the optic axes are parallel to the face p (fig. 95), and make angles of about 58^0 with a normal to the face m. In Oxalic Acid the optic axes are perpendicular to the axis of the zone pe (fig. 96), and make angles of 56^0 with a normal to p. Axinite and Blue Vitriol have each two optic axes, the positions of which with respect to the faces of the crystals have not been well ascertained.

275. The geometrical description of a crystal may be considered complete when we have given—The angles its axes make with each other: the ratios of its parameters: the symbols of the simple forms of which it is a combination, and the symbols of the cleavage forms. We may substitute for the inclinations of the axes and ratios of the parameters certain angles from which the position and mutual inclination of the

faces of a crystal may be more readily calculated. In the pyramidal system, the distance of the pole of (0 0 1) from the pole (1 1 1) or that of (1 0 1), may be given instead of the ratio of the parameters. In the rhombohedral system, the distance between the poles of (1 0 0) and (1 1 1) may be given instead of the inclination of the axes. In the prismatic system, the distance of the pole of (1 1 1) from the poles of two of the three faces (1 0 0), (0 1 0), (0 0 1), may be given instead of the ratios of the parameters. In the oblique prismatic system, the distance between the poles of (1 1 1) and (0 1 0), and the angles which this distance makes with the zone-circles through the pole of (0 1 0) and the poles of (0 0 1) and (1 0 0), may be given instead of the inclination of the axes OZ, OX, and the ratios of the parameters. In the doubly oblique system, the distances between the poles of every two of the three faces (1 0 0), (0 1 0), (0 0 1), and of any two of them from the pole of (1 1 1), may be given instead of the inclinations of the axes and ratios of the parameters.

276. It has been proved in (23) that any given pole, which has an index greater than unity, is the intersection of two zone-circles passing through poles that have lower indices than the given pole. The following table shews how the position of any pole, having no index greater than 7, may be determined by the successive intersections of zone-circles drawn through poles having lower indices, commencing with the poles (1 1 1), ($\bar{1}$ 1 1), (1 $\bar{1}$ 1), (1 1 $\bar{1}$). If we suppose the pole T to be the intersection of the zone-circles PQ, RS, the first column will contain the symbol of T, the second and third columns will contain the symbols of P, Q, and the fourth and fifth those of R, S. When the three indices of T are numerically the same as the three indices in any line of the table, but their order and signs different, the symbols of P, Q, R, S may be found from the symbols given in the table by the application of the rules in (20).

100	$111\ \bar{1}11$	$1\bar{1}1\ 11\bar{1}$
110	$100\ 010$	$111\ 001$
210	$111\ 10\bar{1}$	$100\ 010$
211	$101\ 110$	$111\ 100$
221	$001\ 111$	$100\ 021$
310	$001\ 010$	$10\bar{1}\ 211$
311	$111\ 100$	$1\bar{1}1\ 110$
320	$100\ 010$	$11\bar{1}\ 211$
321	$101\ 110$	$111\ \bar{1}01$
322	$111\ 100$	$120\ 101$
331	$111\ 100$	$120\ 1\bar{1}1$
332	$111\ 001$	$121\ 1\bar{1}0$
410	$100\ 010$	$10\bar{1}\ 212$
411	$111\ 100$	$0\bar{1}1\ 210$
421	$1\bar{1}1\ 110$	$001\ 211$
430	$100\ 010$	$111\ 21\bar{2}$
431	$101\ 110$	$111\ \bar{1}02$
432	$111\ \bar{1}01$	$010\ 211$
433	$111\ 100$	$0\bar{1}1\ 221$
441	$111\ 001$	$201\ 1\bar{2}1$
443	$111\ 001$	$021\ 211$
510	$100\ 010$	$11\bar{2}\ 201$
511	$111\ 100$	$1\bar{1}1\ 210$
520	$100\ 010$	$21\bar{1}\ 102$
521	$100\ 121$	$101\ 11\bar{1}$
522	$111\ 100$	$120\ 201$
530	$100\ 010$	$211\ 11\bar{2}$
531	$111\ 10\bar{1}$	$1\bar{1}1\ 110$
532	$101\ 110$	$111\ 20\bar{1}$
533	$111\ 100$	$1\bar{1}1\ 221$
540	$100\ 010$	$22\bar{1}\ 102$
541	$101\ 110$	$\bar{1}11\ 210$
542	$100\ 021$	$110\ 102$
543	$111\ 10\bar{1}$	$101\ 120$
544	$111\ 100$	$120\ 1\bar{1}2$
551	$111\ 001$	$2\bar{1}1\ 120$
552	$111\ 001$	$321\ 1\bar{1}0$
553	$111\ 001$	$121\ 2\bar{1}0$
554	$111\ 001$	$121\ 2\bar{1}1$
610	$100\ 010$	$201\ 21\bar{2}$
611	$111\ 100$	$310\ 0\bar{1}1$
621	$100\ 021$	$0\bar{1}1\ 201$
631	$001\ 210$	$110\ 2\bar{1}1$
632	$001\ 210$	$110\ 2\bar{1}2$
641	$1\bar{1}1\ 110$	$011\ 210$
643	$010\ 201$	$111\ 20\bar{1}$
650	$100\ 010$	$211\ 0\bar{2}3$
651	$110\ 101$	$\bar{2}11\ 021$
652	$110\ 102$	$211\ 0\bar{2}1$
653	$111\ \bar{1}02$	$010\ 201$
654	$111\ 10\bar{1}$	$211\ 021$
655	$111\ 100$	$0\bar{1}2\ 221$
661	$111\ 001$	$120\ 2\bar{2}1$
665	$111\ 001$	$211\ \bar{2}21$
710	$100\ 010$	$201\ 11\bar{3}$
711	$111\ 100$	$210\ 1\bar{2}1$
720	$100\ 010$	$102\ 31\bar{1}$
721	$100\ 121$	$1\bar{1}1\ 12\bar{1}$
722	$111\ 100$	$1\bar{1}2\ 210$
730	$100\ 010$	$21\bar{1}\ 103$
731	$101\ 210$	$100\ 131$
732	$1\bar{1}1\ 110$	$10\bar{1}\ 112$
733	$111\ 100$	$1\bar{1}1\ 230$
740	$100\ 010$	$102\ 32\bar{1}$
741	$111\ 10\bar{1}$	$121\ \bar{2}11$
742	$100\ 121$	$11\bar{1}\ 211$
743	$110\ 101$	$121\ 2\bar{1}0$
744	$111\ 100$	$102\ 12\bar{3}$
750	$100\ 010$	$1\bar{1}3\ 22\bar{1}$
751	$1\bar{1}1\ 110$	$111\ 121$
752	$110\ 101$	$\bar{1}11\ 310$
753	$111\ 10\bar{1}$	$1\bar{1}1\ 120$
754	$111\ \bar{1}12$	$121\ 1\bar{1}0$
755	$111\ 100$	$12\bar{1}\ 1\bar{1}3$
760	$100\ 010$	$103\ 22\bar{1}$
761	$101\ 110$	$\bar{1}21\ \bar{3}11$

762	110 102	$1\bar{2}1$ 201	772	111 001	130 $2\bar{1}1$
763	100 121	111 $10\bar{3}$	773	111 001	$2\bar{1}0$ 131
764	100 032	111 $21\bar{1}$	774	111 001	121 $3\bar{1}0$
765	111 $10\bar{1}$	121 201	775	111 001	121 $3\bar{1}1$
766	111 100	221 103	776	111 001	121 $2\bar{3}1$
771	111 001	$\bar{1}31$ $\bar{3}21$			

If we commence with (1 1 1), (1 0 0), (0 1 0), (0 0 1), from which the positions of the poles of crystals belonging to the rhombohedral system are most obviously deducible, we must substitute for the first two lines of the table

110	111 001	100 010
$1\bar{1}0$	011 101	100 010
$\bar{1}11$	001 110	100 011

277. Tables for converting the symbol of any form in different systems of crystallographic notation into the equivalent symbol in the notation used in the preceding pages. If the indices, when expressed in numbers, appear as fractions, they must be multiplied by the least common multiple of their denominators.

1. Modified notation of Hauy employed by Mr Brooke, (Enc. Metropolitana, Art. Crystallography). The same table serves also to translate the notation of M. Levy, (Description d'une Collection de Mineraux formée par M. H. Heuland), the indices of which have the same ratios as in the preceding notation, but often differ in magnitude.

Octahedral System.

P	$\{1\,0\,0\}$	$\overset{v}{B}$	$\{v\,1\,0\}$
$\overset{v}{A}$	$\{v\,1\,1\}$	$B_p B_q' B_r$	$\{qr, rp, pq\}$.

Pyramidal System.

M	$\{1\,0\,0\}$	$\overset{v}{B}$	$\{1\,0\,v\}$
P	$\{0\,0\,1\}$	$\overset{v}{G}$	$\{v\,1\,0\}$
$\overset{v}{A}$	$\{1\,1\,v\}$	$B_p B_q' G_r$	$\{qr, rp, pq\}$.
A_v	$\{v\,1\,1\}$		

Rhombohedral System.

P	$\{1\,0\,0\}$	$\overset{v}{E}$ (Levy)	$\{-v\,1\,1\}$
$\overset{v}{A}$	$\{v\,1\,1\}$		
$\overset{v}{E}$	$\{v\,1\,-1\}$	(Prisme hexaèdre; Levy)	
$\overset{v}{O}$	$\{-v\,1\,1\}$	P	$\{1\,1\,1\}$
$\overset{v}{B}$	$\{v\,1\,0\}$	M	$\{2\,-1\,-1\}$
$\overset{v}{D}$	$\{v\,0\,-1\}$	$\overset{v}{B}$	$\{v+2, v-1, v-1\}$
$B_p B_q' B_r'''$	$\{qr, rp, pq\}$	$\overset{v}{A}$	$\{v+3, v, v-3\}$
$D_p D_q' B_r'''$	$\{qr, rp, -pq\}$	A_v	$\{v+3, v, -2v\}$
		$\overset{v}{G}$	$\{v+2, v-1, -1-2v\}$.

Prismatic System.

M	$\{1\,1\,0\}$	A_v	$\{v-1, v+1, 1\}$
P	$\{0\,0\,1\}$	$\overset{v}{H}$	$\{v-1, v+1, 0\}$
$\overset{v}{A}$	$\{0, 2v, 1\}$	$\overset{v}{G}$	$\{v+1, v-1, 0\}$
$\overset{v}{B}$	$\{1\,1\,v\}$	$B_p B_q' H_r$	$\{qr-rp, qr+rp, pq\}$
$\overset{v}{E}$	$\{2v, 0, 1\}$	$B_p B_q'' G_r$	$\{qr+rp, qr-rp, pq\}$.
E_v	$\{v+1, v-1, 1\}$		

Oblique Prismatic System.

M	$\{1\,1\,0\}$	A_v	$\{v+1,\ v-1,\ -1\}$
P	$\{0\,0\,1\}$	E_v	$\{v-1,\ v+1,\ 1\}$
$\overset{v}{O}$	$\{2v,\ 0,\ 1\}$	${}_vE$	$\{1-v,\ 1+v,\ 1\}$
O_v	$\{v+1,\ v-1,\ 1\}$	$\overset{v}{H}$	$\{v+1,\ v-1,\ 0\}$
$\overset{v}{A}$	$\{2v,\ 0,\ 1\}$	$\overset{v}{G}$	$\{v-1,\ v+1,\ 0\}$
$\overset{v}{D}$	$\{1,\ 1,\ v\}$	$D_pD_q'H_r$	$\{rp+qr, rp-qr, pq\}$
$\overset{v}{B}$	$\{1,\ 1,\ -v\}$	$B_pB_q'H_r'$	$\{rp+qr, rp-qr, -pq\}$
$\overset{v}{E}$	$\{0,\ 2v,\ 1\}$	$B_pD_qG_r$	$\{rp-qr, rp+qr, pq\}$

Doubly-oblique Prismatic System.

T	$\{1\,0\,0\}$	$\overset{v}{B}$	$\{1\,0\,-v\}$
M	$\{0\,1\,0\}$	$\overset{v}{F}$	$\{1\,0\,v\}$
P	$\{0\,0\,1\}$	$\overset{v}{H}$	$\{1\,v\,0\}$
$\overset{v}{O}$	$\{v\,v\,1\}$	$\overset{v}{G}$	$\{1\,-v\,0\}$
${}_vO$	$\{v\,1\,v\}$	$D_pF_qH_r$	$\{qr, rp, pq\}$
O_v	$\{1\,v\,v\}$	$C_pB_qH_r'$	$\{qr, rp, -pq\}$
$\overset{v}{D}$	$\{0\,1\,v\}$	$B_pD_qG_r$	$\{-qr, rp, pq\}$
$\overset{v}{C}$	$\{0\,-1\,v\}$	$F_pC_qG_r'$	$\{qr, -rp, pq\}$

The first, second and third indices become respectively negative when E, I and A are substituted for O.

2. Notation of Mohs.

Octahedral System.

H $\{1\,0\,0\}$	B_2 $\{3\,3\,2\}$
O $\{1\,1\,1\}$	C_1 $\{2\,1\,1\}$
D $\{0\,1\,1\}$	C_2 $\{3\,1\,1\}$.
A_1 $\{3\,2\,0\}$	T_1 $\{2\,3\,1\}$
A_2 $\{2\,1\,0\}$	T_2 $\{5\,3\,1\}$
A_3 $\{3\,1\,0\}$	T_3 $\{4\,2\,1\}$
B_1 $\{2\,2\,1\}$	

Pyramidal System.

$[P + \infty\,]$	$\{1\,0\,0\}$
$P - \infty$	$\{0\,0\,1\}$
$P + \infty$	$\{1\,1\,0\}$
$(P + \infty\,)^m$	$\{m\,1\,0\}$
$[(P + \infty\,)^m]$	$\{m + 1, m - 1, 0\}$
P	$\{1\,1\,1\}$
$r\,P + 2n$	$\{r\,2^n, r\,2^n, 1\}$
$r\,P + 2n - 1$	$\{r\,2^n, 0, 1\}$
$r\sqrt{2}\,P + 2n$	$\{r\,2^{n+1}, 0, 1\}$
$r\sqrt{2}\,P + 2n - 1$	$\{r\,2^n, r\,2^n, 1\}$
$(r\,P + 2n)^m$	$\{m\,r\,2^n, r\,2^n, 1\}$
$(r\,P + 2n - 1)^m$	$\{(m + 1)\,r\,2^n, (m - 1)\,r\,2^n, 2\}$
$(r\sqrt{2}\,P + 2n)^m$	$\{(m + 1)\,r\,2^n, (m - 1)\,r\,2^n, 1\}$
$(r\sqrt{2}\,P + 2n - 1)^m$	$\{m\,r\,2^n, r\,2^n, 1\}$.

Rhombohedral System.

$R - \infty$	$\{1\,1\,1\}$	R	$\{1\,0\,0\}$
$R + \infty$	$\{\bar{2}\,1\,1\}$	P^{m}	$\{m + 1, 0, 1 - m\}$
$P + \infty$	$\{0\,\bar{1}\,1\}$	$(P + \infty)^{m}$	$\{3m + 1, -2, 1 - 3m\}$

$rP + n$ is $\{h\,k\,k\}$, where $(h - k) \div (h + 2k) = r2^{n}$. If Q be a pole of any rhombohedron, (Q) its symbol in the notation of Mohs or Naumann; D an adjacent pole of $\{0\,\bar{1}\,1\}$; S a pole of $\{k + l,\ -k,\ -l\}$ adjacent to Q and D; O a pole of $\{1\,1\,1\}$; T the intersection of QD, OS, the symbol of T, in the notation of Mohs or Naumann, will be $(Q)^{m}$, where $m = (k + 2l) \div 3k$.

Prismatic System.

$P + n$	$\{2^{n}, 2^{n}, 1\}$
$\frac{1}{2}(m + 1)\,P + n$	$\{(m + 1)2^{n}, (m + 1)2^{n}, 2\}$
$\frac{1}{2}(m + 1)\,\overline{Pr} + n$	$\{0, (m + 1)2^{n}, 2\}$
$\frac{1}{2}(m + 1)\,\breve{P}r + n$	$\{(m + 1)2^{n}, 0, 2\}$
$(\bar{P} + n)^{m}$	$\{2^{n}, m2^{n}, 1\}$
$(\breve{P} + n)^{m}$	$\{m2^{n}, 2^{n}, 1\}$
$(\overline{Pr} + n)^{m}$	$\{(m - 1)2^{n}, (m + 1)2^{n}, 2\}$
$(\breve{P}r + n)^{m}$	$\{(m + 1)2^{n}, (m - 1)2^{n}, 2\}$.

Oblique Prismatic System.

Let (P) denote the symbol, according to Mohs, of the form $\{h\,k\,l\}$ in the prismatic system. Then $-\frac{(P)}{2}$, $\frac{(P)}{2}$ will be $\{h\,k\,l\}$, $\{h\,k\,\bar{l}\}$ respectively, in the oblique prismatic system. When the form $\{h\,k\,l\}$ has the same number of faces in either system the 2 in the denominator is omitted.

Doubly-oblique Prismatic System.

Let (P) be the symbol, according to Mohs, of the form $\{h\,k\,l\}$ in the prismatic system. Then $-r\frac{(P)}{4}$ will be $\{h\,k\,l\}$; $-l\frac{(P)}{4}\ \{h\,\bar{k}\,l\}$; $r\frac{(P)}{4}\ \{h\,k\,\bar{l}\}$; $l\frac{(P)}{4}\ \{\bar{h}\,k\,l\}$, in the doubly-oblique system. When $\{h\,k\,l\}$ has twice as many faces in one system as in the other the denominator is changed to 2, and when it has the same number of faces in either system it is omitted.

3. Notation of Naumann.

Octahedral System.

$\infty O \infty$	$\{1\,0\,0\}$	mOm	$\{m\,1\,1\}$
O	$\{1\,1\,1\}$	mO	$\{m\,m\,1\}$
∞O	$\{0\,1\,1\}$	mOn	$\{m, mn, n\}$.
∞On	$\{n\,1\,0\}$		

Pyramidal System.

$\infty P \infty$	$\{1\,0\,0\}$	mP	$\{m\,m\,1\}$
$0P$	$\{0\,0\,1\}$	$mP\infty$	$\{m\,0\,1\}$
∞P	$\{1\,1\,0\}$	∞Pn	$\{n\,1\,0\}$
P	$\{1\,1\,1\}$	mPn	$\{m, mn, n\}$.
$P\infty$	$\{1\,0\,1\}$		

Rhombohedral System.

$0R$	$\{1\,1\,1\}$	R^{m}	$\{m+1,\ 0,\ 1-m\}$
∞R	$\{2\,\bar{1}\,\bar{1}\}$	$2R^{m}$	$\{m,\ 1,\ -m\}$
$\infty P2$	$\{0\,\bar{1}\,1\}$	$\infty P'$	$\{-m,\ m-1,\ 1\}$
R	$\{1\,0\,0\}$	mR	$\{2m+1, 1-m, 1-m\}$.

Prismatic System.

P	$\{1\,1\,1\}$	$m\bar{P}n$	$\{m, mn, n\}$
$0P$	$\{0\,0\,1\}$	$\bar{P}n$	$\{1\,n\,n\}$
∞P	$\{1\,1\,0\}$	$m\breve{P}n$	$\{mn, m, n\}$
mP	$\{m\,m\,1\}$	$\breve{P}n$	$\{n\,1\,n\}$.

Oblique Prismatic System.

$0P$	$\{0\,0\,1\}$	$-mPn$	$\{-mn, m, n\}$
P	$\{1\,1\,1\}$	Pn	$\{n\,1\,n\}$
mP	$\{m\,m\,1\}$	(mPn)	$\{m, mn, n\}$
$-mP$	$\{-m\,m\,1\}$	(Pn)	$\{1\,n\,n\}$.
mPn	$\{mn, m, n\}$		

Doubly-oblique Prismatic System.

P' $\{1\,1\,1\}$; $'P$ $\{\bar{1}\,1\,1\}$; $P_{,}$ $\{1\,1\,\bar{1}\}$; $_{,}P$ $\{1\,\bar{1}\,1\}$. The symbols of the other faces are derived from P', $'P$, $P_{,}$, $_{,}P$ as in the prismatic system.

4. Notation of Weiss.

Octahedral System.

$$\frac{1}{h}a : \frac{1}{k}a : \frac{1}{l}a \qquad \{h\,k\,l\}.$$

Pyramidal System.

$$\frac{1}{h}a : \frac{1}{k}a : \frac{1}{l}c \qquad \{h\,k\,l\}.$$

Rhombohedral System.

$$\left.\begin{array}{c} \dfrac{c}{h+k+l} \\ \dfrac{a}{h+k-2l} : \dfrac{a}{h-2k+l} : \dfrac{a}{-2h+k+l} \end{array}\right\} \quad \{h\,k\,l\}.$$

Prismatic System.

$$\frac{1}{h}a : \frac{1}{k}b : \frac{1}{l}c \qquad \{h\,k\,l\}.$$

CHAPTER X.

DRAWING CRYSTALS AND PROJECTIONS.

278. The figures of crystals are usually projections, upon a plane, of the edges in which their faces intersect, by lines parallel to a given straight line. Since the lengths of the edges of a crystal are not subject to any known law, the edges in the figure need not be either equal or proportional to the projections of the edges of any given crystal. In the rhombohedral system the plane of projection is generally parallel to one axis, and makes equal angles with each of the other two axes. In the remaining systems the plane of projection is generally parallel to two of the axes. The following rules for drawing crystals will not require any demonstration.

279. To draw the axes of a crystal.

When a plane of projection is parallel to the axes OZ, OX, draw ZOZ', XOX' (fig. 135), making with each other an angle equal to the angle between the corresponding axes of the crystal, and any line YOY' making finite angles with ZOZ', XOX'. Then, the three lines XOX', YOY', ZOZ' will represent the three axes of the crystal.

If in OZ, OX, we take OC, OA proportional to the parameters c, a respectively, and OB of any magnitude, OA, OB, OC will represent the parameters of the crystal.

In the rhombohedral system, make the angle ROC (fig. 136) equal to the angle between a normal to (1 1 1) and either of the axes of the crystal. Draw CRS perpendicular to OR, making $RS = \frac{1}{2} CR$. Through S draw ASB making any angle with CS, and in ASB take A, B at any equal distances

from S. Then, the three straight lines OA, OB, OC will represent the axes, and the distances OA, OB, OC will represent the parameters.

When the position of the plane of projection with respect to the axes is arbitrary, the axes may be represented by any three lines meeting in one point.

280. To draw lines parallel to the intersections of the plane of the face $(h\,k\,l)$ with the planes through the axes.

Let OA, OB, OC (fig. 135.) represent the parameters of the crystal. Take OH, OK, OL respectively proportional to $\frac{1}{h}OA$, $\frac{1}{k}OB$, $\frac{1}{l}OC$. Then KL, LH, HK will be parallel to the lines in which the plane of the face $(h\,k\,l)$ intersects the planes YOZ, ZOX, XOY.

When one of the indices h, k, l is zero, the corresponding axis will be parallel to two of the intersections.

281. Let the sides of the triangles HKL, PQR (fig. 137.) be parallel to the intersections of the planes of two given faces with the planes YOZ, ZOX, XOY. Let the intersections with YOZ, ZOX, XOY meet in U, V, W respectively. Then a line through any two of the three points U, V, W, will pass through the third, and will be parallel to the projection of the edge formed by the intersection of the two given faces.

If we draw lines, by the above method, parallel to the projections of all the edges of a crystal, meeting each other in the order in which the edges meet, the figure thus obtained will be the figure of the crystal.

282. To draw the axes of a twin crystal.

Construct fig. 138. so that OU, OV, OW may be proportional to the parts of the axes of either crystal cut off by the twin plane, and VOW, WOU, UOV equal to the angles YOZ, ZOX, XOY respectively. From the points O draw lines perpendicular to the adjacent sides of the triangle UVW, meeting

in T, and draw the lines UTu, VTv, WTw. Let OX, OY, OZ (fig. 139.) represent the axes of one of the crystals; U, V, W the points in which they are intersected by the twin face. Divide two of the sides of UVW into segments proportional to the corresponding sides of UVW (fig. 138.), and let the lines through the points of division and the opposite angles intersect in T. Then OT represents a perpendicular on the plane UVW. Take $OO' = 2OT$. Then $O'U$, $O'V$, $O'W$ will represent the axes of the second crystal. Let $(u\,v\,w)$ be the symbol of the twin face. Take $O'A' = u \,.\, OU$, $O'B' = v \,.\, O'V$, $O'C' = w \,.\, O'W$; then $O'A'$, $O'B'$, $O'C'$ will represent the parameters of the second crystal. The axes and parameters having been projected, the faces of the two crystals may be drawn by the rules already given.

283. No representation of a crystal is capable of exhibiting the relative positions of its faces, the zones which they form, and the kind of symmetry with which they are arranged, so clearly as the figure of a sphere having the poles of the faces of the crystal laid down upon its surface. This method of representing a crystal, which was invented by Professor Neumann of Königsberg, possesses the additional advantage of enabling us to investigate all the geometrical properties of crystals by spherical trigonometry alone, without the aid of either solid or analytical geometry.

When the figure of the sphere is either its stereographic or gnomonic projection, many problems of crystallography may be very expeditiously solved by simple geometrical constructions. On this account the following investigation of the principal properties of the stereographic and gnomonic projections has been given.

284. In the stereographic projection, points and circles upon the surface of a sphere are referred to the plane of a great circle of the sphere, called the primitive, by lines drawn to one of the poles of the great circle. To an eye placed in this pole of the primitive, the projection of any point will be the picture of that point on the plane of the primitive.

285. Let O (fig. 140) be the center of a sphere which is to be projected stereographically; E, C the poles of the primitive, the eye being at E; P', Q' any two points on the surface of the sphere. The straight line EC meets the plane of the primitive in O; $\therefore$ O is the projection of C. Draw the straight lines EP', EQ' meeting the plane of the primitive in P, Q; $\therefore$ P, Q are the projections of P', Q'. Let r be the radius of the sphere. Then $OP = r \tan OEP = r \tan \frac{1}{2} CP'$. In like manner $OQ = r \tan \frac{1}{2} CQ'$.

The angles QOP, $Q'CP'$ are manifestly equal.

A straight line drawn from E to any point in the great circle CP' will meet the plane of the primitive in OP. Hence a great circle passing through the poles of the primitive is projected into a straight line passing through the center of the primitive.

286. Let Q' be any point in a circle of which P' is the farther pole; P, Q the projections of P', Q'. Then

$$\cos P'Q' = \cos P'C \cos Q'C + \sin P'C \sin Q'C \cos Q'CP'$$

$$= \cos P'C \frac{r^2 - QO^2}{r^2 + QO^2} + 2 \sin P'C \frac{r \,.\, QO}{r^2 + QO^2} \cos QOP;$$

whence,

$$(\cos P'Q' + \cos P'C)\, QO^2 - 2r \sin P'C \cos QOP \,.\, QO$$
$$+ r^2 (\cos P'Q' - \cos P'C) = 0.$$

Therefore Q is a point in a circle having its center in OP.

287. Let the given circle meet CP' in M', N'; and let M, N be the projections of M', N'. Then MN will be a diameter of the projection of the circle.

Let K be the center of the circle MQN. Then

$$2KQ = r \tan \tfrac{1}{2} (P'Q' + CP') + r \tan \tfrac{1}{2} (P'Q' - CP'),$$
$$2KO = r \tan \tfrac{1}{2} (P'Q' + CP') - r \tan \tfrac{1}{2} (P'Q' - CP').$$

When Q' is a point in a great circle, $P'Q'$ is a quadrant.

Therefore $KQ = r \sec CP'$, $\quad KO = r \tan CP'$.

When Q' is a point in a small circle the pole of which is in the primitive, CP' is a quadrant. Therefore

$$KQ = r \tan P'Q', \quad KO = r \sec P'Q'.$$

A circle passing through E, the place of the eye, will manifestly be projected into a straight line.

288. To draw the projection of a great circle through two given points.

Let Q be the most distant of the two points P, Q (fig. 141); O the center of the projection. Draw OE perpendicular to OQ meeting the primitive in E; EQ meeting the primitive in q; $q\,O$ meeting the primitive in s; Es meeting QO in S. A circle through QRS will be the projection of a great circle. For QS is the projection of an arc equal to qs; $\therefore$ Q, S are the projections of opposite points of the two extremities of a diameter of the sphere. Therefore the circle projected into QRS is a great circle.

289. Having given the projection of a great circle; to find the projection of its pole.

Let GMH (fig. 142) be the projection of a great circle, meeting the primitive in G,H, and, therefore, GH a diameter of the primitive. Through O, the center of the primitive, draw MO perpendicular to GH. Draw GM meeting the primitive in m. Make mp a quadrant; and draw Gp meeting MO in P. Then P will be the point required. For MP is the projection of a quadrant mp, and G, H are the poles of the circle projected into MP. Therefore GMH, P are the projections of a great circle and its pole.

290. If a great circle and its pole be projected into GQH and P (fig 143); and if straight lines PQ, PR be drawn meeting the primitive in q, r, the arc qr will be equal to the arc projected into RQ. For PQq, PRr are the projections of small circles passing through the pole of the circle projected

into GQH and through the place of the eye, which is one pole of GqH. But two small circles drawn through the poles of two equal circles manifestly intercept equal arcs of the equal circles. Therefore rq is equal to the arc projected into RQ.

291. To find the angle between two great circles; having given their projections.

Let GR, LR (fig. 144) be the projections of two great circles intersecting in R. Let the poles of the circles be projected into P, Q. Draw the straight lines RP, RQ meeting the primitive in p, q. The angle between the circles projected into GR, LR is measured by pq. For R is the projection of the pole of the great circle projected into PQ. Therefore pq measures the distance between P, Q, the poles of the circles projected into GR, LR, and therefore it measures the angle between the circles which are projected into GR, LR.

292. Let O be the center of the primitive MN (fig. 145); MQ the projection of a great circle MQ'; K its center; C the farther pole of the primitive; CQ' a great circle meeting MQ' in Q' and the primitive in N. Then the straight line OQ will be the projection of CQ'; $OQ = r \tan \frac{1}{2} CQ'$; $KQ = r \sec Q'MN$; $KO = r \tan Q'MN$. The spherical triangle MNQ' is right-angled at N. Whence

$$\sin KQO = \frac{KO}{KQ} \sin KOL = \sin Q'MN \cos MN = \cos MQ'N.$$

$$LO = KO \sin MN = r \tan Q'MN \sin MN = r \tan NQ'.$$

Hence, when the arc CQ' and the angle $MQ'C$ are given, if we make

$$OQ = r \tan \tfrac{1}{2} CQ'; \quad OL = r \cot CQ'; \quad OQK = 90^0 - MQ'C,$$

and draw LK perpendicular to ON; the circle MQ described round K as a center, with the radius KQ, will be the projection of MQ'.

293. Let SQ be the projection of any other great circle $S'Q'$ passing through Q', R its center. Then

$$OQR = 90^0 - S'Q'C.$$

But $OQK = 90^0 - MQ'C$; $\therefore$ $KQR = MQ'S'$.

Therefore the angle between any two great circles is equal to the angle which their projections make with each other at the point in which they intersect.

294. When a crystal belongs to the octahedral system, the projection of the sphere, upon which its poles are laid down, may have either the zone-circle through two adjacent poles of $\{0\,1\,1\}$, or the zone-circle through two adjacent poles of $\{1\,0\,0\}$, for its primitive. When the crystal belongs to the pyramidal or prismatic system, it will be found convenient to take the zone-circle through the poles $(1\,0\,0)$, $(0\,1\,0)$ for the primitive. When the crystal belongs to the rhombohedral system, the primitive should be the zone-circle containing the poles of $\{0\,\bar{1}\,1\}$. When the crystal belongs to the oblique-prismatic system, the primitive should be the zone-circle through $(0\,0\,1)$ and $(1\,0\,0)$. When the crystal belongs to the doubly-oblique system, any zone-circle may be taken for the primitive.

295. To draw the stereographic projection of the poles of a crystal of Axinite (fig. 100), having given $mp = 45^0.12'$, $pf = 44^0.43'$, $mx = 49^0.32'$, $my = 79^0.24'$, $fx = 64^0.57'$.

Let the zone-circle mp (fig. 101) be taken for the primitive; and let O be its center, r its radius. Make $mp = 45^0.12'$, $pf = 44^0.53'$, and draw diameters mOm', pOp', fOf'. In Of take $OK = r \sec 64^0.57'$, and in Om take $OL = r \sec 49^0.32'$, and with centers K, L and radii $Kx = r \tan 64^0.57'$, $Lx = r \tan 49^0.32'$ describe circles intersecting in x. Draw the circles mxm', pxp', fxf'. Draw OM perpendicular to Om, meeting mxm' in M; $m'M$ meeting the primitive in M'; take $M'N'$ a quadrant, and draw $m'N'$ meeting OM in N. In mpm' take $mY = 79^0.24'$, and draw the straight line NY meeting mxm' in y. Draw pyp', and fyf' meeting pxp' in v. Draw

mvm' meeting fxf' in t, and pyp' in w; fwf' meeting mxm' in c, and pxp' in n, and ptp' meeting mxm' in s. Draw OT perpendicular to Ot meeting the primitive in T; draw TU perpendicular to tT meeting Ot in U, and draw the circle Uyt meeting mpm' in e, and fwf' in o. Draw eve' meeting pyp' in q; fsf' meeting mvm' in l; plp' meeting mxm' in r, and mom' meeting fyf' in g. Then m, p, x, &c. will be the projections of the poles of the faces m, p, x, &c.

296. In the gnomonic projection of the sphere, points upon the surface of the sphere are referred to a plane touching the sphere by lines drawn through them from the center of the sphere. To an eye placed in the center of the sphere, the projection of any point will be the picture of the point upon the plane of projection.

297. Let O (fig. 146) be the center of a sphere; C the point in which the sphere touches the plane of projection, and which is termed the center of the projection; P, Q any two points on the sphere. Draw straight lines OP', OQ' meeting the plane of projection in P, Q. Therefore P, Q are the projections of P', Q'. Let r be the radius of the sphere. Then $CP = r \tan CP'$, $CQ = r \tan CQ'$, and $QCP = Q'CP'$.

The plane of every great circle passes through O, and intersects the plane of projection in a straight line. Hence the projection of a great circle is a straight line.

Let PQ be the projection of a great circle $P'Q'$ having its poles in the great circle CP'. Then, since the planes CPQ, OPQ are perpendicular to the plane CPO, their intersection PQ will be perpendicular to CPO, and, therefore, PQ will be perpendicular to CP.

298. Let O (fig. 147) be the center of the sphere; C the center of the projection; $Q'R'$ an arc of a great circle; QR its projection. Draw ECB perpendicular to QR. Make $DB = CO$, $EB = CD$. QR is perpendicular to BE and BO, and $BE = CD = BO$. Whence $QER = QOR$. Therefore QER measures the arc $Q'R'$.

Hence we can measure the arc of a great circle projected into a given straight line; or cut off a portion of a given straight line, which shall be the projection of a given arc of a great circle.

299. Let C (fig. 148.) be the center of the projection; O the center of the sphere; CQ, PQ the projections of two great circles CQ', $P'Q'$; CP' perpendicular to CQ', and, therefore, ECP perpendicular to CQ.

$$\tan PQC = \frac{CP}{CQ} = \frac{\tan CP'}{\tan CQ'} = \cos CQ' \tan CQ'P'.$$

Take $CE = CO$. Draw CF perpendicular to QE, and make $CG = CF$. Then, $CG = CQ \cos CQ'$, $\tan PQC = \cos CQ'$ $\tan CGP$. Therefore $CGP = CQ'P'$.

Hence we may either find the angle between two great circles of which the projections are given; or, having given the angle between two great circles, the projection of one of them may be drawn through a given point in the projection of the other.

When CQ' is small, the above method of comparing the angle between two great circles with the angle between their projections is not so accurate as the following.

300. Let C (fig. 149) be the center of the projection; CQ, PQ the projections of two great circles CQ', $P'Q'$. Draw HQ perpendicular to CQ, and make HQ equal to the radius of the sphere. Make $KQ = CH$; bisect HK in L, and let PQ meet HK in P.

$$\tan PQC = \tan K \frac{KP}{HP} = \frac{HQ}{KQ}\,\frac{KP}{HP} = \cos CQ' \,\frac{KP}{HP}.$$

$$\text{Hence } \tan Q' = \frac{KP}{HP} = \frac{HL - LP}{HL + LP};$$

$$\therefore \frac{LP}{HL} = \frac{1 - \tan Q'}{1 + \tan Q'} = \tan (45^0 - Q').$$

If $l^0 = 90^0 - CQ'$, $\dfrac{HQ}{KQ} = \sin l^0$, $\therefore HQ = \dfrac{KH.\sin l^0}{\sqrt{\{1 + (\sin l^0)^2\}}}$.

For the purpose of facilitating the above construction sectors are provided with a scale called "inclination of meridians," in which the distance between the divisions $0^0, m^0 = c\{1 - \tan(45^0 - m^0)\}$, and with a scale called "latitudes," in which the distance between the divisions $0^0, l^0$

$$= \frac{2c \sin l^0}{\sqrt{\{1 + (\sin l^0)^2\}}}.$$

Let rad. sphere $= CQ \tan l^0$. Draw QH perpendicular to CQ, making $QH = (0^0, l^0)$ of the line of latitudes. With center H, and radius $HK = (0^0, 90^0)$ of the line of inclination of meridians, describe a circle cutting CQ in K. Then, it will be easily seen that, if $KM = (0^0, m^0)$ of the line of inclination of meridians, MQ will be the projection of a great circle that makes an angle of m^0 with the great circle projected into CQ.

301. The gnomonic projection may be used with advantage in projecting the poles of a crystal belonging to either of the three systems in which the three axes make right angles with each other. The plane of projection is most conveniently situated when it meets the three axes at equal distances from their intersection. The axes YZ, ZX, XY will then be projected into the three sides of an equilateral triangle.

When the crystal belongs to the rhombohedral system, the plane of projection may be parallel to a face of $\{1\,1\,1\}$, or of $\{2\,\bar{1}\,\bar{1}\}$. When the crystal belongs to the oblique prismatic system, the plane of projection may be parallel to the faces of $\{0\,1\,0\}$; and when it belongs to the doubly-oblique system, the plane of projection may be parallel to any face.

302. To draw the gnomonic projection of the poles of a crystal of Topaz (fig. 87) on a plane meeting the axes at equal distances from their intersections; having given $ru = 22^0.15'$, $rl = 43^0.26'$, $rm = 62^0.10'$, $pn = 43^0.30'$, $py = 62^0.13'$, $po = 45^0.27'$.

Let f be the pole (0 1 0). Then r, f, p will coincide with X, Y, Z, and the arcs joining the poles of r, f, p will be projected into the equilateral triangle r, f, p (fig. 134). Let C be the middle point of the triangle. Draw fC, pC meeting pr, rf in M, N. Let O be the centre of the sphere. Then OC will be perpendicular to the plane rfp. $Or = Of$, and $rOf = 90^\circ$, whence $ON = Nr$. With centre C and radius Nr describe a circle cutting Nr in Q. CN is common to the triangles QNC, OCN, $QC = NO$, $QNC = OCN$, therefore $OC = QN$. In Np take $NR = Nr$, and make $uRr = 32^\circ.15'$, $lRr = 43^\circ.26'$, $mRr = 62^\circ.10'$. In Mf take $MS = Nr$, and make $nSp = 43^\circ.30'$, $ySp = 62^\circ.13'$. Draw CT perpendicular to pm. In Tp take $TU = OC$. In TC take $TV = UC$, and make $oVp = 45^\circ.27'$. Let no meet pl in x and pf in i. Let rx meet pm in s; and let fo meet pr in t. Then p, r, m, &c. will be the projections of the poles of the faces p, r, m, &c.

THE END.

PLATE I.

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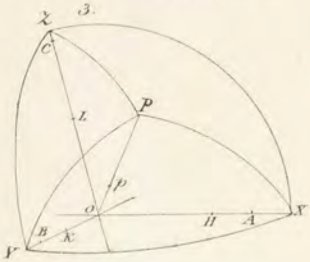

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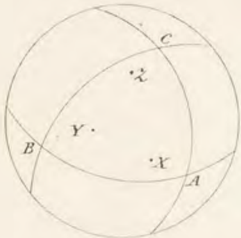

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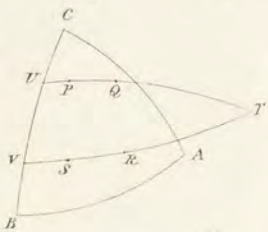

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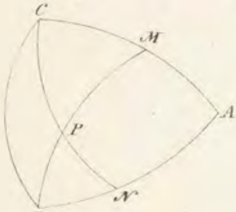

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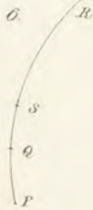

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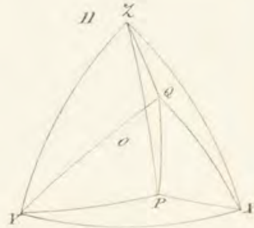

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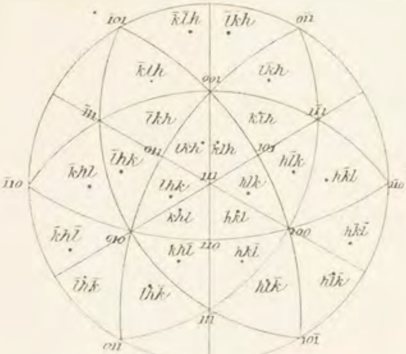

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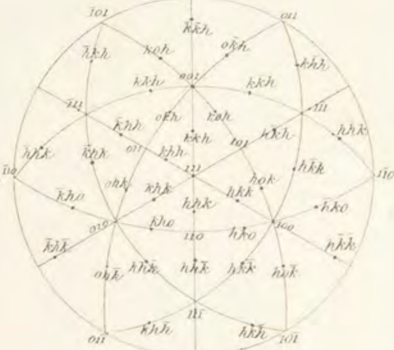

J. Grieve, Lithog.

PLATE II.

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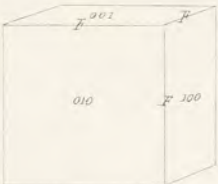

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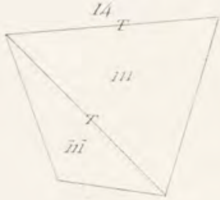

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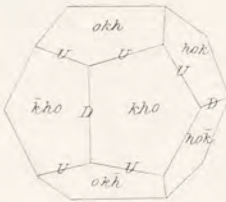

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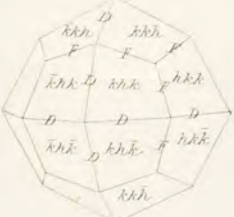

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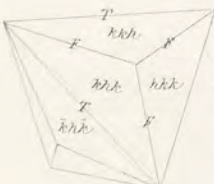

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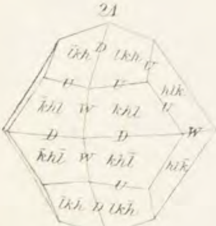

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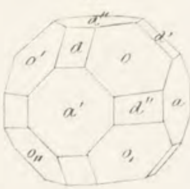

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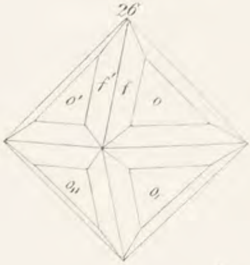

J. Grieve, Lincoy.

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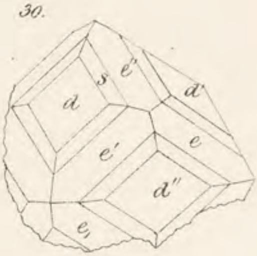

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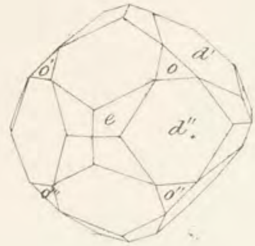

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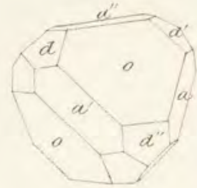

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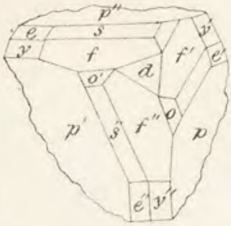

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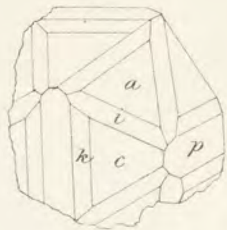

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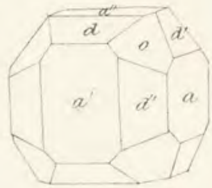

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J. Grieve, Zincog

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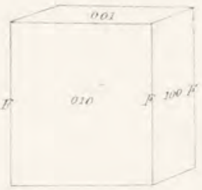

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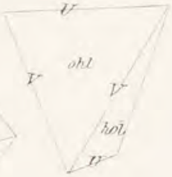

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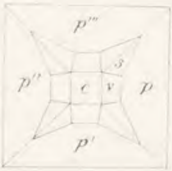

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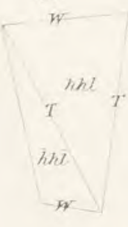

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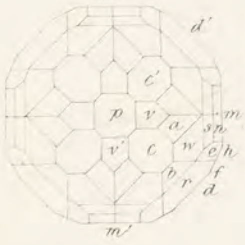

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PLATE V.

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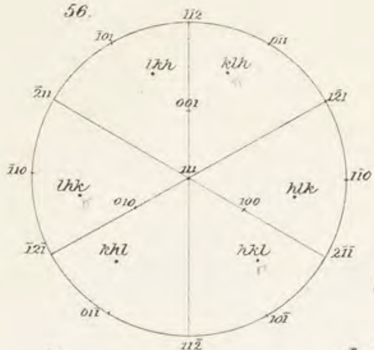

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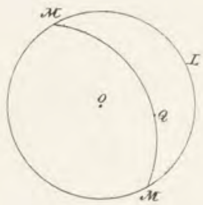

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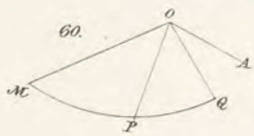

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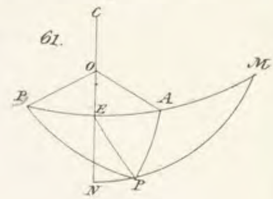

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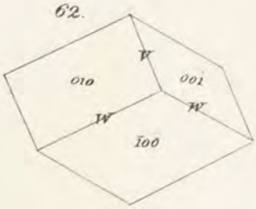

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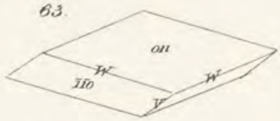

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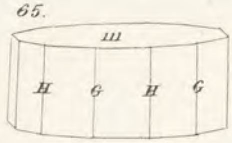

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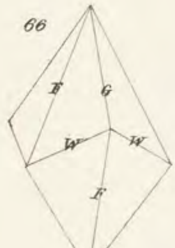

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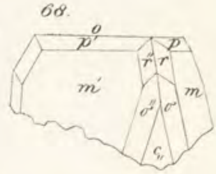

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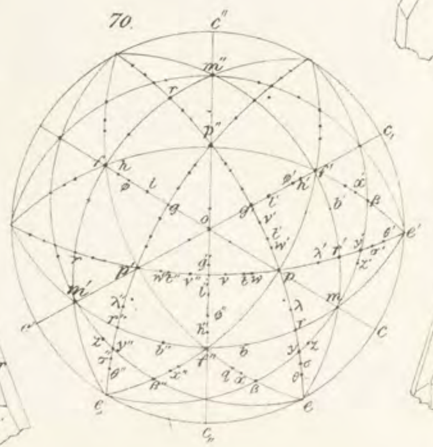

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PLATE VI.

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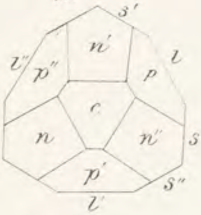

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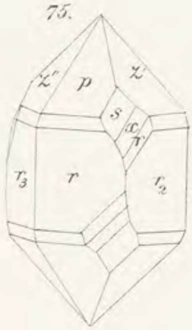

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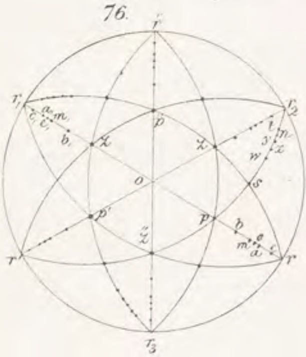

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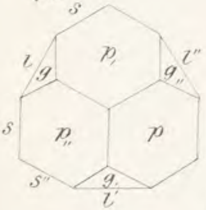

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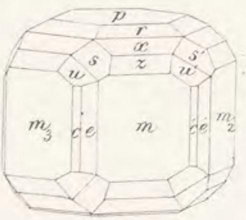

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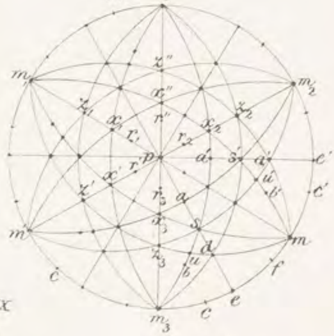

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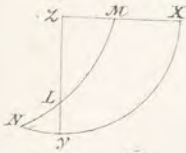

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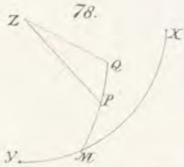

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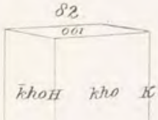

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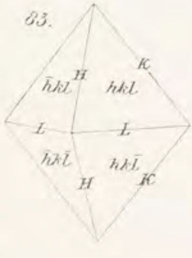

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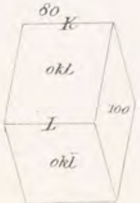

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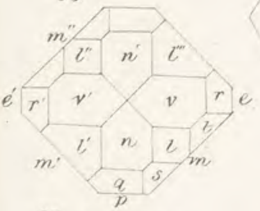

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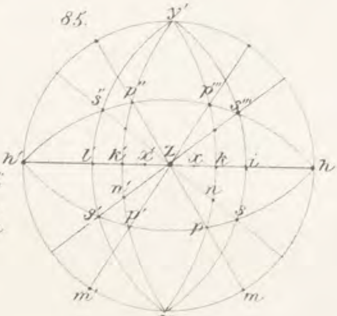

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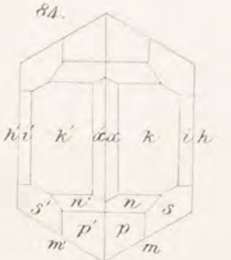

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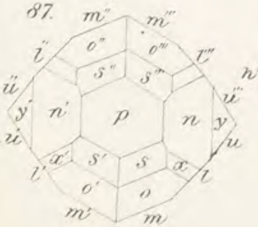

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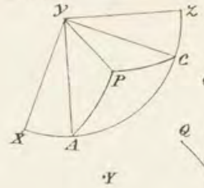
88.



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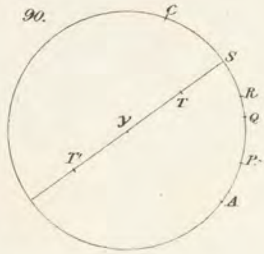

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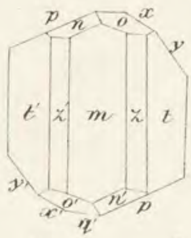

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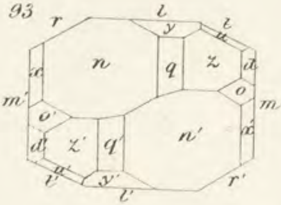

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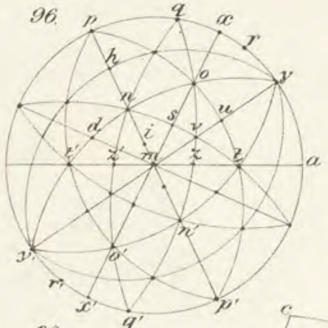

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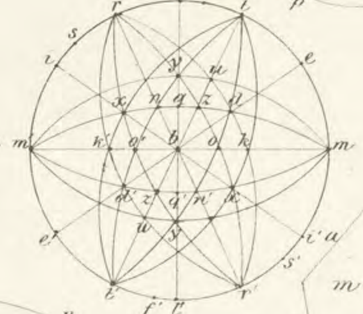

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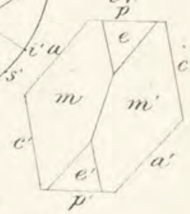

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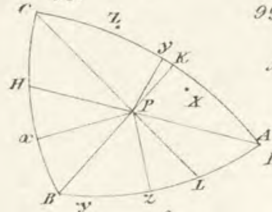

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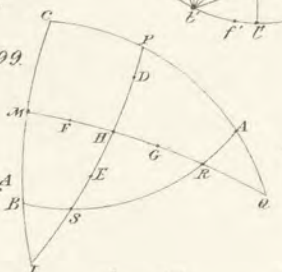

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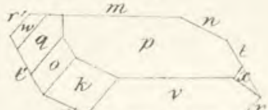

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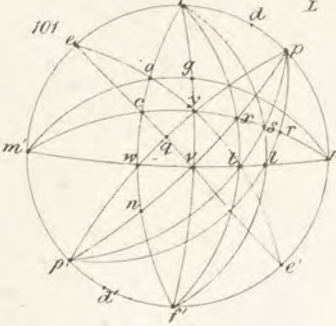

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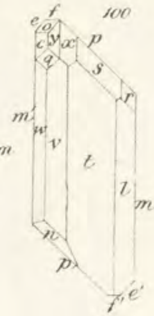

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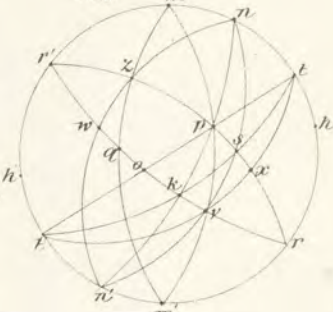

PLATE VIII

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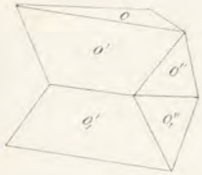

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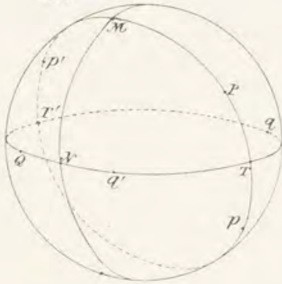

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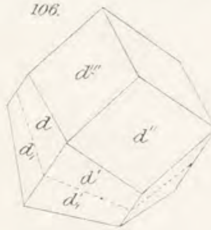

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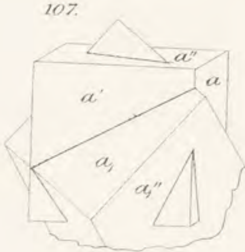

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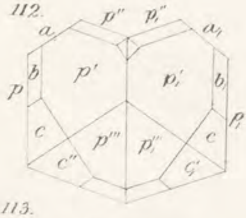

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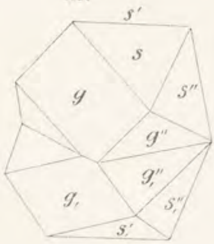

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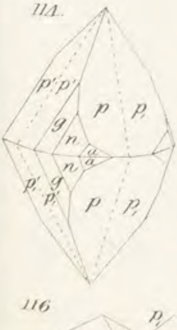

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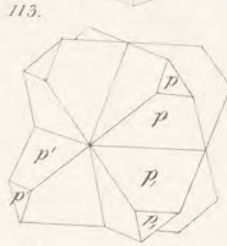

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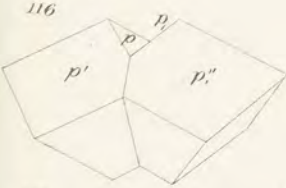

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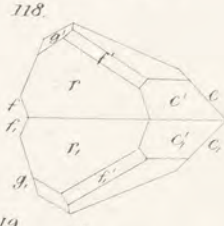

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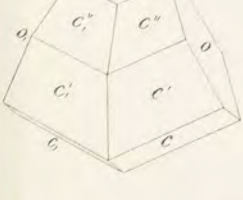

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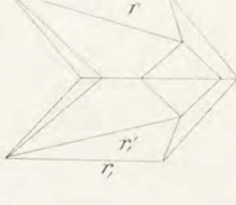

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PLATE IX.

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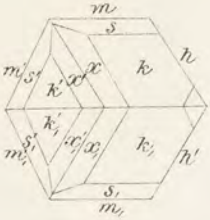

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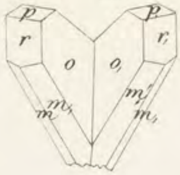

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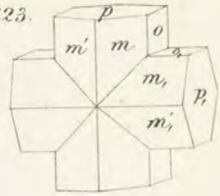

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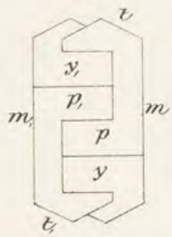

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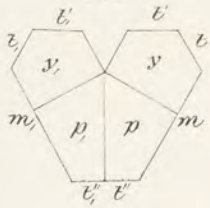

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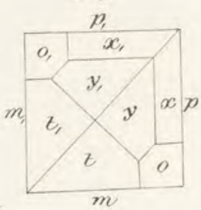

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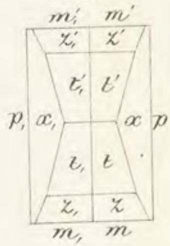

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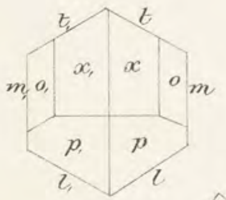

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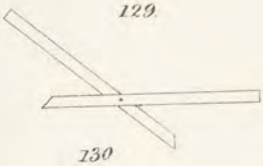

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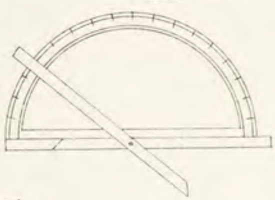

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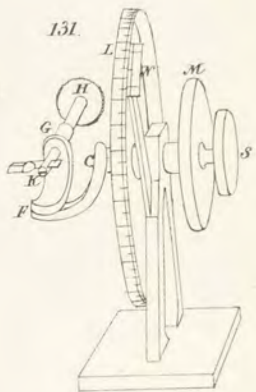

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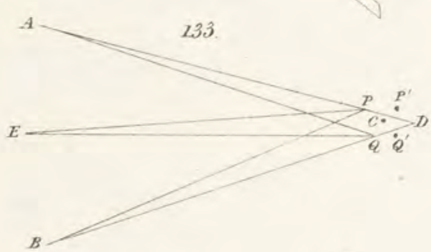

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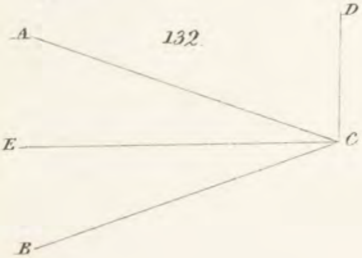

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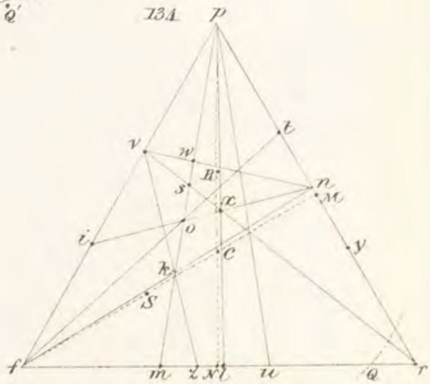

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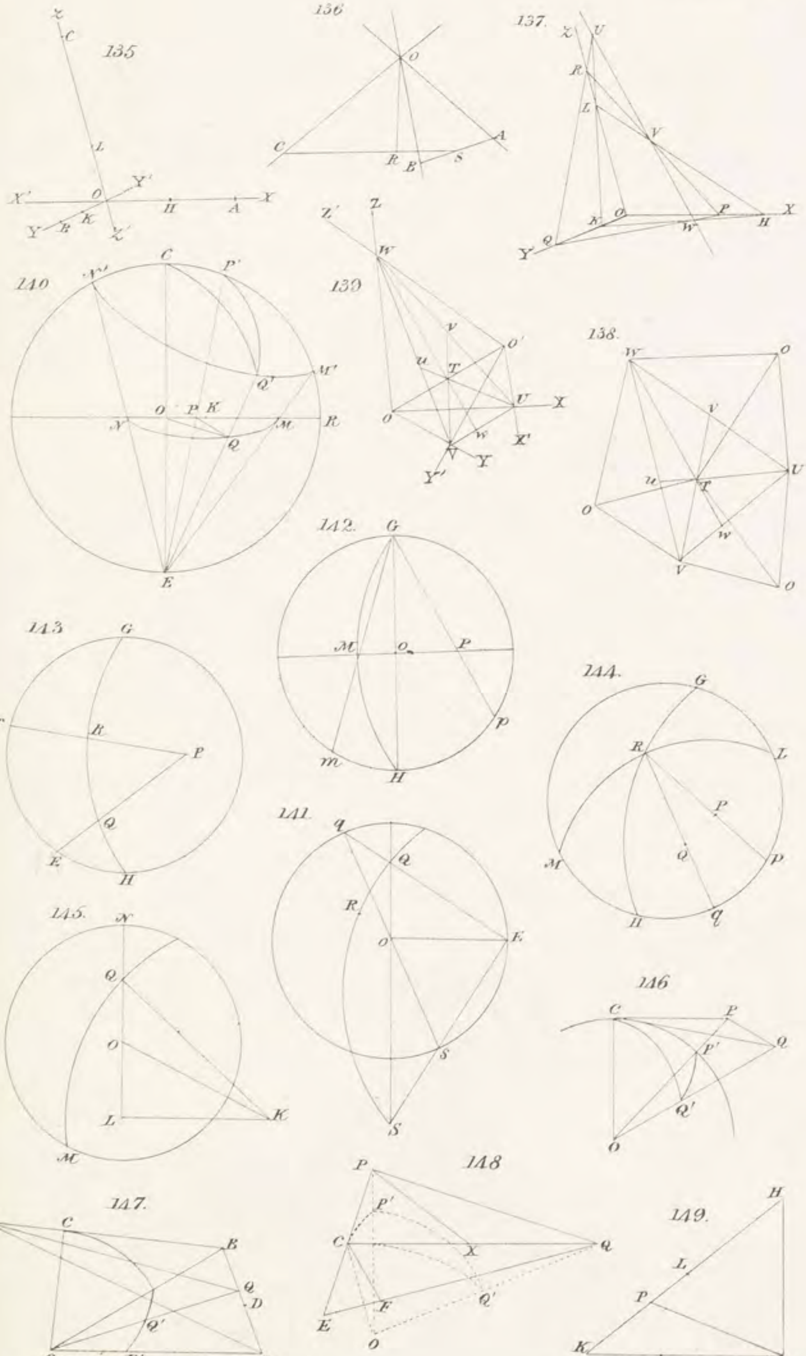
PLATE X.



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